

COMPUTING SIMPLE MULTIPLE ZEROS OF POLYNOMIAL SYSTEMS

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ABSTRACT. Given a polynomial system f associated with a simple multiple zero x of multiplicity μ , we give a computable lower bound on the minimal distance between the simple multiple zero x and other zeros of f . If x is only given with limited accuracy, we propose a numerical criterion that f is certified to have μ zeros (counting multiplicities) in a small ball around x . Furthermore, for simple double zeros and simple triple zeros whose Jacobian is of normalized form, we define modified Newton iterations and prove the quantified quadratic convergence when the starting point is close to the exact simple multiple zero. For simple multiple zeros of arbitrary multiplicity whose Jacobian matrix may not have a normalized form, we perform unitary transformations and modified Newton iterations, and prove its non-quantified quadratic convergence and its quantified convergence for simple triple zeros.

CONTENTS

1. Introduction	2
2. Preliminaries	7
2.1. Local Dual Space	7
2.2. Simple Multiple Zeros	8
2.3. Normalized Form	9
3. Local Separation Bound and Cluster Location	10
3.1. Simple Triple Zeros	10
3.2. Simple Multiple Zeros	17
3.3. Re-examining Double Simple Zeros	25
4. Modified Newton Iterations	28
4.1. γ -theorem for Simple Double Zeros	28
4.2. γ -theorem for Simple Triple Zeros	34
4.3. Simple Multiple Zeros	41
References	57

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1. INTRODUCTION

Consider an ideal I_f generated by a polynomial system $f = \{f_1, \dots, f_n\}$, where $f_i \in \mathbb{C}[X_1, \dots, X_n]$. An isolated zero of multiplicity μ for f is a point $x \in \mathbb{C}^n$ such that

- (1) $f(x) = 0$,
- (2) there exists a ball $B(x, r)$ of radius $r > 0$ such that $B(x, r) \cap f^{-1}(0) = \{x\}$,
- (3) $\mu = \dim(\mathbb{C}[X]/Q_{f,x})$,

where

$$B(x, r) := \{y \in \mathbb{C}^n : \|y - x\| < r\},$$

and $Q_{f,x}$ is an isolated primary component of the ideal I_f whose associate prime is

$$m_x = (X_1 - x_1, \dots, X_n - x_n).$$

In [8], based on Rouché's Theorem [1], the condition (3) is replaced by

- (3a) a generic analytic g sufficiently close to f has m simple zeros in $B(x, r)$.

We recall α -theory below according to [2] and refer to [40, 38, 39, 41, 37, 44, 17] for more details. Let $Df(x)$ denote the Jacobian matrix of f at x . Suppose $Df(x)$ is invertible, x is called a simple zero of f . The Newton's iteration is defined by

$$(1.1) \quad N_f(x) = x - Df(x)^{-1}f(x).$$

Shub and Smale [37] defined

$$(1.2) \quad \gamma(f, x) = \sup_{k \geq 2} \left\| Df(x)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}},$$

where $D^k f$ denotes the k -th derivative of f which is a symmetric tensor whose components are the partial derivatives of f of order k , $\|\cdot\|$ denotes the classical operator norm.

According to [2, Theorem 1], if

$$(1.3) \quad \|z - x\| \leq \frac{3 - \sqrt{7}}{2\gamma(f, x)},$$

then Newton's iterations starting at z will converge quadratically to the simple zero x .

If y is another zero of f , according to [2, Corollary 1], we have

$$(1.4) \quad \|y - x\| \geq \frac{5 - \sqrt{17}}{4\gamma(f, x)},$$

which separates the simple zero x from other zeros of f .

Furthermore, according to [2, Theorem 2], if only a system f and a point x are given such that

$$(1.5) \quad \alpha(f, x) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.157671,$$

where $\alpha(f, x) = \beta(f, x)\gamma(f, x)$ and

$$\beta(f, x) = \|x - N_f(x)\| = \|Df(x)^{-1}f(x)\|,$$

then Newton's iterations starting at x will converge quadratically to a simple zero ξ of f and

$$\|x - \xi\| \leq 2\beta(f, x).$$

In [8], Dedieu and Shub gave quantitative results for simple double zeros satisfying $f(x) = 0$ and

- (A) $\dim \ker Df(x) = 1$,
- (B) $D^2 f(x)(v, v) \notin \text{im} Df(x)$,

where $\ker Df(x)$ is spanned by a unit vector $v \in \mathbb{C}^n$. They generalized the definition of γ (1.2) to

$$(1.6) \quad \gamma_2(f, x) = \max \left(1, \sup_{k \geq 2} \left\| A(f, x, v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}} \right),$$

where

$$(1.7) \quad A(f, x, v) = Df(x) + \frac{1}{2} D^2 f(x)(v, \Pi_v),$$

is a linear operator which is invertible at the simple double zero x , and Π_v denotes the Hermitian projection onto the subspace $[v] \subset \mathbb{C}^n$.

In [8, Theorem 1], Dedieu and Shub also presented a lower bound for separating simple double zeros x from the other zeros y of f ,

$$(1.8) \quad \|y - x\| \geq \frac{d}{2\gamma_2(f, x)^2},$$

where $d \approx 0.2976$ is a positive real root of

$$(1.9) \quad \sqrt{1 - d^2} - 2d\sqrt{1 - d^2} - d^2 - d = 0.$$

Remark 1. *There are two typos in the statements of [8, Theorem 1] and [8, Lemma 4]: 1) the degree of γ_2 in (1.8) is 2 instead of 1 in [8, Theorem 1]; 2) the coefficient of the second $\sqrt{1 - d^2}$ in (1.9) is $-2d$ instead of $-d$ in [8, Lemma 4].*

In [8, Theorem 4], Dedieu and Shub showed that if the following criterion is satisfied at a given point x and a given vector v

$$(1.10) \quad \|f(x)\| + \|Df(x)v\| \frac{d}{4\gamma_2(f, x, v)^2} < \frac{d^3}{32\gamma_2^4 \|B(f, x, v)^{-1}\|},$$

then f has two zeros in the ball of radius

$$(1.11) \quad \frac{d}{4\gamma_2(f, x)^2},$$

around x . Let us set

$$B(f, x, v) = A(f, x, v) - L,$$

where $L(v) = Df(x)v$, $L(w) = 0$ for $w \in v^\perp$, and

$$(1.12) \quad \gamma_2(f, x) = \max \left(1, \sup_{k \geq 2} \left\| B(f, x, v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}} \right).$$

Our Contributions. We generalize Dedieu and Shub's quantitative results about simple double zeros to simple multiple zeros whose Jacobian matrix has corank one.

Let us recall our previous work on computing simple multiple zeros. Suppose x is an isolated singular zero of f satisfying $\dim \ker Df(x) = 1$. Let $\mathcal{D}_{f,x}$ denote the local dual space of an ideal $I_f = (f_1, \dots, f_n)$ at x :

$$(1.13) \quad \mathcal{D}_{f,x} := \{\Lambda \in \mathfrak{D}_x \mid \Lambda(g) = 0, \forall g \in I_f\},$$

where $\mathfrak{D}_x = \text{span}_{\mathbb{C}}\{\mathbf{d}_x^\alpha\}$ is the \mathbb{C} -vector space generated by differential functionals \mathbf{d}_x^α of order $\alpha \in \mathbb{N}^n$, see (2.1). Let μ denote the multiplicity, then starting from $\Lambda_0 = 1$, and

$$\Lambda_1 = d_1 + a_{1,2}d_2 + \cdots + a_{1,n}d_n,$$

where d_1, \dots, d_n are the first order differential functionals, we can construct

$$\Lambda_k = \Delta_k + a_{k,2}d_2 + \cdots + a_{k,n}d_n,$$

incrementally for $k = 2, \dots, \mu-1$ by formulas (2.6) and (2.8), s.t. $\{\Lambda_0, \Lambda_1, \dots, \Lambda_{\mu-1}\}$ is a closed basis of the local dual space $\mathcal{D}_{f,x}$ [23, 25]. The method is efficient since the size of matrices involved in the computation is bounded by n .

We generalize the definition of simple double zeros in [8]. A simple multiple zero x of f satisfies $f(x) = 0$ and

- (A) $\dim \ker Df(x) = 1$,
- (B) $\Delta_k(f) \in \text{im } Df(x)$, for $k = 2, \dots, \mu-1$,
- (C) $\Delta_\mu(f) \notin \text{im } Df(x)$.

Without loss of generality, we can assume that $Df(x)$ has a normalized form:

$$(1.14) \quad Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix},$$

where $D\hat{f}(x)$ is the nonsingular Jacobian matrix of polynomials $\hat{f} = \{f_1, \dots, f_{n-1}\}$ with respect to variables $\hat{X} = \{X_2, \dots, X_n\}$. We will show in Section 2.3 that it is always possible to perform unitary transformations to f and variables X to obtain an equivalent polynomial system whose Jacobian matrix at the singular solution has the normalized form (1.14). This normalization step is similar to the reduction to one variable technique used in [10].

If x is a simple multiple zero of f of multiplicity μ and $Df(x)$ has the normalized form (1.14), then it is clear that

$$\Delta_k(f) \in \text{im } Df(x) \Leftrightarrow \Delta_k(f_n) = 0,$$

and the above (B) and (C) conditions can be simplified to

- (B) $\Delta_k(f_n) = 0$, for $k = 2, \dots, \mu-1$,
- (C) $\Delta_\mu(f_n) \neq 0$.

For a simple multiple zero x satisfying conditions (A),(B) and (C), we generalized the definition of γ_2 in (1.6) to

$$(1.15) \quad \gamma_\mu = \gamma_\mu(f, x) = \max(\hat{\gamma}_\mu, \gamma_{\mu,n}),$$

where

$$(1.16) \quad \hat{\gamma}_\mu = \hat{\gamma}_\mu(f, x) = \max \left(1, \sup_{k \geq 2} \left\| D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!} \right\|^{\frac{1}{k-1}} \right),$$

and

$$(1.17) \quad \gamma_{\mu,n} = \gamma_{\mu,n}(f, x) = \left(1, \sup_{k \geq 2} \left\| \frac{1}{\Delta_\mu(f_n)} \cdot \frac{D^k f_n(x)}{k!} \right\|^{\frac{1}{k-1}} \right),$$

where $D^k \hat{f}(x)$ for $k \geq 2$ denote the partial derivatives of \hat{f} of order k with respect to X_1, X_2, \dots, X_n evaluated at x . We generalize main results in [8] to simple multiple zeros of higher multiplicities.

Firstly, in Theorem 5, we present a lower bound for separating simple multiple zeros x of multiplicity $\mu \geq 2$ from another zero y of f ,

$$(1.18) \quad \|y - x\| \geq \frac{d}{2\gamma_\mu(f, x)^\mu},$$

where d is a positive real root of a univariate polynomial $p(d)$ defined in (3.19). The explicit formulas of $p(d)$ for multiplicity 2 and 3 are given in (3.24) and (3.6). In Section 3.3, we also compare our local separation bound for simple double zeros with the one given in [8]. Although the smallest positive real root $d \approx 0.2865$ of (3.6) is smaller than $d \approx 0.2976$ given in [8], our value of γ_2 could be smaller too. Therefore, for some examples (see Example 1), our local separation bound still could be larger than the one given in [8].

Secondly, we define

$$(1.19) \quad H_1 = \begin{pmatrix} \frac{\partial \hat{f}(x)}{\partial X_1} & 0 \\ \frac{\partial f_n(x)}{\partial X_1} & \frac{\partial f_n(x)}{\partial X} \end{pmatrix},$$

and tensors

$$(1.20) \quad H_k = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ \Delta_k(f_n) & 0 \end{pmatrix} \underbrace{\mathbf{0}_{n \times \cdots \times n \times (n-1)}}_k \end{pmatrix}, \quad 2 \leq k \leq \mu - 1,$$

and polynomials

$$g(X) = f(X) - f(x) - \sum_{1 \leq k \leq \mu-1} H_k(X - x)^k.$$

Let \mathcal{A} be an invertible matrix

$$(1.21) \quad \mathcal{A} = \begin{pmatrix} \sqrt{2}D\hat{f}(x) & 0 \\ 0 & \frac{1}{\sqrt{2}}\Delta_\mu(f_n) \end{pmatrix}.$$

In Theorem 8, we show that if

$$(1.22) \quad \|f(x)\| + \sum_{1 \leq k \leq \mu-1} \|H_k\| \left(\frac{d}{4\gamma_\mu(g, x)^\mu} \right)^k < \frac{d^{\mu+1}}{2(4\gamma_\mu(g, x)^\mu)^\mu} \|\mathcal{A}^{-1}\|,$$

then f has μ zeros (counting multiplicities) in the ball of radius $\frac{d}{4\gamma_\mu(g, x)^\mu}$ around x .

Thirdly, we design modified Newton iterations and extend the γ -theorem for simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form (1.14). Given an approximate zero z of f with an associated exact simple double zero ξ , we show in Theorem 9 that when

$$\|z - \xi\| < \frac{0.0318}{\gamma_2(f, \xi)^2},$$

after k times of the modified Newton iteration defined in Algorithm 4.1, we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2} \right)^{2^k - 1} \|z - \xi\|.$$

Similarly, for an approximate zero z of f with an associated exact simple triple zero ξ , we show in Theorem 10 that when

$$\|z - \xi\| < \frac{0.0154}{\gamma_3(f, \xi)^3},$$

after k times of the modified Newton iteration defined in Algorithm 4.2, we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

Finally, for simple multiple zeros whose Jacobian matrix is of corank one but it does not have a normalized form (1.14), we apply the unitary transformations defined in Section 2.3 to obtain an equivalent polynomial system whose Jacobian matrix at the approximate simple multiple zeros z has the normalized form approximately:

$$(1.23) \quad Df(z) = \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix},$$

where σ_n is its smallest singular value and Σ_{n-1} is a nonsingular diagonal matrix. Then we perform the Newton iteration to refine the last $n-1$ variables. After the Newton iteration, we need to perform the unitary transformations again to ensure that the Jacobian matrix at the refined approximate solution satisfies (1.23). We define the modified Newton iterations based on our previous work in [24, Algorithm 1] to refine the first variable. We show in Theorem 11, for

$$\hat{\gamma}_\mu(f, z) \|z - \xi\| < \frac{1}{2},$$

the refined singular solution $N_f(z)$ returned by Algorithm 4.3 satisfies

$$\|N_f(z) - \xi\| = O(\|z - \xi\|^2).$$

Furthermore, we show in Theorem 12 that for an approximate zero z of a system f associated to a simple triple zero ξ , when

$$\|z - \xi\| < \frac{0.0098}{\gamma_3(f, \xi)^3},$$

after k times of the modified Newton iterations defined in Algorithm 4.3, we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

It is clear that the proof of Theorem 12 can be generalized to give a quantitative quadratic convergence of Algorithm 4.3 for simple multiple zeros of arbitrary higher multiplicities.

Related Works. Yakoubsohn [46] extended α -theory to clusters of univariate polynomials and provided an algorithm to compute clusters of univariate polynomials [47]. Giusti, Lecerf, Salvy and Yakoubsohn [9] presented point estimate criteria for cluster location of analytic functions in the univariate case. They provided bounds on the diameter of the cluster which contains μ zeros (counting multiplicities) of f . They proposed an algorithm based on the Schröder iterations for approximating clusters and provided a stopping criterion which guarantees the algorithm converges to the cluster quadratically. In [10], they further generalized their results to locate and approximate clusters of zeros of analytic maps of embedding dimension one in the multivariate case. They reduced this particular multivariate case to univariate case via implicit theorem and deflation techniques. We are inspired by their technique of reduction to one variable but we try to avoid the use of implicitly known univariate analytic function. Dedieu and Shub [8] gave explicitly a lower bound

for separating simple double zeros x from other zeros of f (1.8) and a criterion (1.10) depending only on the approximate solution which guarantees the existence of a cluster of two zeros. Based on our previous work [23, 25] on computing multiplicity structure of simple multiple zeros, we generalize Dedieu and Shub's results and deal with simple multiple zeros with higher multiplicities. The proof of the non-quantified quadratic convergence [24, Theorem 3.16] of Algorithm 1 in [24] has also been simplified.

There are other approaches to compute isolated multiple zeros or zero clusters, e.g., corrected Newton methods [33, 5, 6, 7, 14, 15, 34, 35, 29], deflation techniques [32, 48, 31, 20, 4, 21, 22, 45, 3, 36, 24, 27, 11, 26, 18, 16]. We refer to [10, 16] for excellent introductions of previous works on approximating multiple zeros.

Structure of the Paper. In Section 2, we recall some notations and show how to compute incrementally a closed basis of the local dual space of I_f at a given multiple root x of corank 1 and multiplicity μ . In Section 3, we begin with explaining how to extend main results in [8] to simple triple zeros. We present a lower bound for separating simple triple zeros from other zeros of f and an explicit criterion that guarantees the existence of a cluster of three zeros of f around the approximate singular solution x . Then we generalize these results to simple multiple zeros with arbitrary higher multiplicities. We also compare our local separation bound for simple double zeros with the one given in [8]. In Section 4, we design modified Newton iterations and extend the γ -theory for simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form. For a simple multiple zero of arbitrary large multiplicity whose Jacobian matrix does not have a normalized form, we perform unitary transformations and modified Newton iterations, and show non-quantified quadratic convergence of new algorithm. Furthermore, we show its quantified convergence for simple triple zeros.

2. PRELIMINARIES

2.1. Local Dual Space. Let $\mathbf{d}_x^\alpha : \mathbb{C}[X] \rightarrow \mathbb{C}$ denote the differential functional defined by

$$(2.1) \quad \mathbf{d}_x^\alpha(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x), \quad \forall g \in \mathbb{C}[X],$$

where $x \in \mathbb{C}^n$ and $\alpha = [\alpha_1, \dots, \alpha_n] \in \mathbb{N}^n$. We have

$$(2.2) \quad \mathbf{d}_x^\alpha((X - x)^\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Let I_f be an ideal generated by a polynomial system $f = \{f_1, \dots, f_n\}$, where $f_i \in \mathbb{C}[X_1, \dots, X_n]$. The local dual space of I_f at a given isolated singular solution x is a subspace $\mathcal{D}_{f,x}$ of $\mathfrak{D}_x = \text{span}_{\mathbb{C}}\{\mathbf{d}_x^\alpha\}$ such that

$$(2.3) \quad \mathcal{D}_{f,x} = \{\Lambda \in \mathfrak{D}_x \mid \Lambda(g) = 0, \forall g \in I_f\}.$$

When the evaluation point x is clear from the context, we write $d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ instead of \mathbf{d}_x^α for simplicity.

Let $\mathcal{D}_{f,x}^{(k)}$ be the subspace of $\mathcal{D}_{f,x}$ with differential functionals of orders bounded by k , we define

$$(1) \text{ breadth } \kappa = \dim \left(\mathcal{D}_{f,x}^{(1)} \setminus \mathcal{D}_{f,x}^{(0)} \right),$$

$$(2) \text{ depth } \rho = \min \left(\left\{ k \mid \dim \left(\mathcal{D}_{f,x}^{(k+1)} \setminus \mathcal{D}_{f,x}^{(k)} \right) = 0 \right\} \right),$$

$$(3) \text{ multiplicity } \mu = \dim \left(\mathcal{D}_{f,x}^{(\rho)} \right).$$

If x is an isolated singular solution of f , then $1 \leq \kappa \leq n$ and $\rho < \mu < \infty$.

Let us introduce a morphism $\Phi_\sigma : \mathfrak{D}_x \rightarrow \mathfrak{D}_x$ which is an anti-differentiation operator defined by

$$\Phi_\sigma(d_1^{\alpha_1} \cdots d_n^{\alpha_n}) = \begin{cases} d_1^{\alpha_1} \cdots d_\sigma^{\alpha_\sigma-1} \cdots d_n^{\alpha_n}, & \text{if } \alpha_\sigma > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Computing a closed basis of the local dual space is done essentially by matrix-kernel computations based on the stability property of $\mathcal{D}_{f,x}$ [28, 30, 42, 4]:

$$(2.4) \quad \forall \Lambda \in \mathcal{D}_{f,x}^{(k)}, \quad \Phi_\sigma(\Lambda) \in \mathcal{D}_{f,x}^{(k-1)}, \quad \sigma = 1, \dots, n.$$

2.2. Simple Multiple Zeros. In this paper, we deal with simple multiple zeros satisfying $f(x) = 0$, $\dim \ker Df(x) = 1$. It is also called breadth one singular zero in [4] as

$$(2.5) \quad \dim(\mathcal{D}_{f,x}^{(k)} \setminus \mathcal{D}_{f,x}^{(k-1)}) = 1, \quad k = 1, \dots, \rho, \quad \rho = \mu - 1.$$

Therefore, the local dual space of I_f at a given isolated simple singular solution x is

$$\mathcal{D}_{f,x} = \text{span}_{\mathbb{C}}\{\Lambda_0, \Lambda_1, \dots, \Lambda_{\mu-1}\},$$

where $\deg(\Lambda_k) = k$ and $\Lambda_0 = 1$. Suppose $\Lambda_1 = a_{1,1}d_1 + \cdots + a_{1,n}d_n$, without loss of generality, we assume $a_{1,1} = 1$. Let $\Psi_\sigma : \mathfrak{D}_x \rightarrow \mathfrak{D}_x$ be a differential operator defined by

$$\Psi_\sigma(d_1^{\alpha_1} \cdots d_n^{\alpha_n}) = \begin{cases} d_\sigma^{\alpha_\sigma+1} \cdots d_n^{\alpha_n}, & \text{if } \alpha_1 = \cdots = \alpha_{\sigma-1} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 2, \dots, \mu - 1$, by the stability property, we have

$$(2.6) \quad \begin{cases} \Phi_1(\Lambda_k) = a_{1,1}\Lambda_{k-1} + \cdots + a_{k-1,1}\Lambda_1 + a_{k,1}\Lambda_0, \\ \vdots \\ \Phi_n(\Lambda_k) = a_{1,n}\Lambda_{k-1} + \cdots + a_{k-1,n}\Lambda_1 + a_{k,n}\Lambda_0. \end{cases}$$

Let $a_{1,1} = 1$, $a_{k,1} = 0$ ($k = 2, \dots, n$), the system (2.6) has a unique solution $\Lambda_k = \Delta_k + a_{k,2}d_2 + \cdots + a_{k,n}d_n$, where

$$(2.7) \quad \Delta_k = \sum_{\sigma=1}^n \Psi_\sigma(a_{1,\sigma}\Lambda_{k-1} + \cdots + a_{k-1,\sigma}\Lambda_1),$$

and $a_{k,2}, \dots, a_{k,n}$ are determined by solving the linear system obtained from $\Lambda_k(f_i) = 0, i = 1, \dots, n$:

$$(2.8) \quad \begin{pmatrix} d_2(f_1) & \cdots & d_n(f_1) \\ \vdots & \ddots & \vdots \\ d_2(f_n) & \cdots & d_n(f_n) \end{pmatrix} \begin{pmatrix} a_{k,2} \\ \vdots \\ a_{k,n} \end{pmatrix} = - \begin{pmatrix} \Delta_k(f_1) \\ \vdots \\ \Delta_k(f_n) \end{pmatrix}.$$

We refer to [23, 24, 25] for the justification of above arguments.

The following definition generalizes the simple double zero in [8].

Definition 1. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, where $f_i \in \mathbb{C}[X]$ and suppose $f(x) = 0$. Then x is a simple zero of multiplicity μ for f if

- (A) $\dim \ker Df(x) = 1$,
- (B) $\Delta_k(f) \in \text{im } Df(x)$, for $k = 2, \dots, \mu - 1$,
- (C) $\Delta_\mu(f) \notin \text{im } Df(x)$.

In fact, for $\mu = 2$, suppose $\ker Df(x) = \text{span}_{\mathbb{C}}\{v\}$ with $\|v\| = 1$, then $\Lambda_1(f) = Df(x) \cdot v = v_1 d_1(f) + \dots + v_n d_n(f)$ and

$$\begin{aligned}
 \Delta_2(f) &= \sum_{\sigma=1}^n \Psi_{\sigma}(v_{\sigma} \Lambda_1)(f) \\
 &= \sum_{\sigma=1}^n \Psi_{\sigma}(v_{\sigma}(v_1 d_1 + \dots + v_n d_n))(f) \\
 &= \sum_{i>j} v_i v_j d_i d_j(f) + \sum_i v_i^2 d_i^2(f) \\
 &= \frac{1}{2} D^2 f(x)(v, v).
 \end{aligned}$$

Hence, the condition $\Delta_2(f) \notin \text{im } Df(x)$ is equivalent to $D^2 f(x)(v, v) \notin \text{im } Df(x)$, the condition given for the simple double zero in [8].

2.3. Normalized Form. We show below that it is always possible to perform unitary transformations to obtain an equivalent polynomial system whose Jacobian matrix at the simple multiple zero has a normalized form.

Definition 2. For a polynomial function $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, where $f_i \in \mathbb{C}[X_1, \dots, X_n]$, $Df(x)$ has a normalized form if

$$(2.9) \quad Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix},$$

$D\hat{f}(x)$ is the nonsingular Jacobian matrix of polynomials $\hat{f} = \{f_1, \dots, f_{n-1}\}$ with respect to variables X_2, \dots, X_n .

Let $Df(x) = U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \cdot V^*$ be the singular value decomposition of $Df(x)$ of corank 1, where $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ are unitary matrices, V^* is the Hermitian transpose of V , and Σ_{n-1} is a nonsingular diagonal matrix. We can always assume that $Df(x)$ has a normalized form (2.9). Otherwise, let $g = U^* \cdot f(W \cdot X)$, where $W = (v_n, v_1, \dots, v_{n-1})$ is also a unitary matrix. Suppose x is a simple multiple zero of f of multiplicity μ , then W^*x is a simple multiple zero of g of multiplicity μ and the Jacobian matrix of g at W^*x has a normalized form:

$$\begin{aligned}
 Dg(W^*x) &= U^* \cdot Df(x) \cdot W \\
 &= U^* \cdot U \cdot \Sigma \cdot V^* \cdot W \\
 &= \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & \Sigma_{n-1} \\ 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Furthermore, suppose y is another zero of f , then W^*y is another zero of g , and the Euclidean distance between zeros x and y does not change under the unitary

transformation:

$$\|W^*x - W^*y\| = \|W^*(x - y)\| = \|x - y\|.$$

If x is a simple multiple zero of multiplicity μ for f and $Df(x)$ has the normalized form (2.9). Then we have

$$\text{im } Df(x) = \text{im} \begin{pmatrix} D\hat{f}(x) \\ 0 \end{pmatrix},$$

and

$$\Delta_k(f) \in \text{im } Df(x) \Leftrightarrow \Delta_k(f_n) = 0.$$

The (B)(C) conditions can be simplified to check only the last polynomial f_n :

(B) $\Delta_k(f_n) = 0$, for $k = 2, \dots, \mu - 1$,

(C) $\Delta_\mu(f_n) \neq 0$.

The linear system (2.8) for getting the values of $a_{k,2}, \dots, a_{k,n}$ can be simplified to:

$$(2.10) \quad \begin{pmatrix} d_2(f_1) & \cdots & d_n(f_1) \\ \vdots & \ddots & \vdots \\ d_2(f_{n-1}) & \cdots & d_n(f_{n-1}) \end{pmatrix} \begin{pmatrix} a_{k,2} \\ \vdots \\ a_{k,n} \end{pmatrix} = - \begin{pmatrix} \Delta_k(f_1) \\ \vdots \\ \Delta_k(f_{n-1}) \end{pmatrix}.$$

3. LOCAL SEPARATION BOUND AND CLUSTER LOCATION

We begin with explaining how to extend main results in [8] to simple triple zeros. Then we generalize these results to simple multiple zeros with arbitrary higher multiplicities. We also compare our local separation bound for simple double zeros with the one given in [8].

3.1. Simple Triple Zeros. Let x be a simple triple zero of f and $Df(x)$ has the normalized form (2.9), i.e.

$$\frac{\partial f_i(x)}{\partial X_1} = 0, \quad \frac{\partial f_n(x)}{\partial X_i} = 0, \quad 1 \leq i \leq n$$

and

$$\Delta_2(f_n) = 0, \quad \Delta_3(f_n) \neq 0.$$

Let $\Lambda_0 = 1$, $\Lambda_1 = d_1$. By (2.7), we have

$$\Delta_2 = \sum_{\sigma=1}^n \Psi(a_{1,\sigma} \Lambda_1) = d_1^2,$$

and

$$\Lambda_2 = d_1^2 + a_{2,2}d_2 + \cdots + a_{2,n}d_n,$$

where $a_{2,2}, \dots, a_{2,n}$ satisfy

$$\begin{pmatrix} a_{2,2} \\ \vdots \\ a_{2,n} \end{pmatrix} = -D\hat{f}(x)^{-1} \begin{pmatrix} \Delta_2(f_1) \\ \vdots \\ \Delta_2(f_{n-1}) \end{pmatrix} = -D\hat{f}(x)^{-1} \begin{pmatrix} d_1^2(f_1) \\ \vdots \\ d_1^2(f_{n-1}) \end{pmatrix},$$

since $d_2(f_n) = \dots = d_n(f_n) = 0$, $\Delta_2(f_n) = 0$ and the Jacobian $D\hat{f}(x)$ of polynomials $\hat{f} = \{f_1, \dots, f_{n-1}\}$ with respect to variables $\hat{X} = \{X_2, \dots, X_n\}$ is invertible. Moreover, since $a_{1,1} = 1$, $a_{2,1} = 0$, we have

$$\begin{aligned} \Delta_3 &= \sum_{\sigma=1}^n \Psi_{\sigma}(a_{1,\sigma}\Lambda_2 + a_{2,\sigma}\Lambda_1) \\ &= \Psi_1(\Lambda_2) + \sum_{\sigma=1}^n \Psi_{\sigma}(a_{2,\sigma}d_1) \\ &= d_1^3 + a_{2,2}d_1d_2 + \dots + a_{2,n}d_1d_n. \\ &= d_1^3 + (d_1d_2, \dots, d_1d_n) \cdot \left(-D\hat{f}(x)^{-1}\right) \cdot \begin{pmatrix} d_1^2(f_1) \\ \vdots \\ d_1^2(f_{n-1}) \end{pmatrix}. \end{aligned}$$

For simplicity, we use the following equivalent conditions

$$(3.1) \quad \Delta_2(f_n) = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} = 0,$$

$$(3.2) \quad \Delta_3(f_n) = \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1^3} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \neq 0.$$

The definition of γ_3 has been given in (1.15):

$$\gamma_3 = \gamma_3(f, x) = \max(\hat{\gamma}_3, \gamma_{3,n}),$$

where

$$\hat{\gamma}_3 = \hat{\gamma}_3(f, x) = \max \left(1, \sup_{k \geq 2} \left\| D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!} \right\|^{\frac{1}{k-1}} \right),$$

and

$$\gamma_{3,n} = \gamma_{3,n}(f, x) = \max \left(1, \sup_{k \geq 2} \left\| \frac{1}{\Delta_3(f_n)} \cdot \frac{D^k f_n(x)}{k!} \right\|^{\frac{1}{k-1}} \right).$$

For two nonzero vectors $a, b \in \mathbb{C}^n$, we define their angle by

$$(3.3) \quad d_P(a, b) = \arccos \frac{|\langle a, b \rangle|}{\|a\| \cdot \|b\|}.$$

Let y be another vector in \mathbb{C}^n and $y \neq x$ and define

$$w = y - x = \begin{pmatrix} \zeta \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}.$$

Let $\varphi = d_P(v, y - x)$, $v = (1, 0, \dots, 0)^T$, then we have

$$|\zeta| = \|w\| \cos \varphi, \quad \|\eta\| = \|w\| \sin \varphi.$$

For $k \geq 2$, we use $D^k \hat{f}(x)$ to denote the partial derivatives of \hat{f} of order k with respect to X_1, X_2, \dots, X_n . We generalize main results in [8] to simple triple zeros.

Lemma 1. *If $\hat{\gamma}_3(f, x)\|w\| \leq \frac{1}{2}$, then*

$$\left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| \geq \|w\| \sin \varphi - 2\hat{\gamma}_3(f, x)\|w\|^2.$$

Proof. By Taylor's expansion of $\hat{f}(y)$ at x , and $\frac{\partial \hat{f}(x)}{\partial X_1} = 0$, we have

$$\hat{f}(y) = \hat{f}(x) + D\hat{f}(x)\eta + \sum_{k \geq 2} \frac{D^k \hat{f}(x)(y-x)^k}{k!}.$$

Noticing that $\hat{f}(x) = 0$ and $D\hat{f}(x)$ is invertible, we have

$$\eta = D\hat{f}(x)^{-1}\hat{f}(y) - \sum_{k \geq 2} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!}.$$

By the triangle inequality, we have

$$\begin{aligned} \|w\| \sin \varphi = \|\eta\| &\leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + \sum_{k \geq 2} \left\| D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!} \right\| \|y-x\|^k \\ &\leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + \sum_{k \geq 2} \hat{\gamma}_3(f, x)^{k-1} \|w\|^k \\ &\leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + 2\hat{\gamma}_3(f, x)\|w\|^2, \end{aligned}$$

where the last inequality comes from the assumption that $\hat{\gamma}_3(f, x)\|w\| \leq \frac{1}{2}$. \square

Let

$$(3.4) \quad \mathcal{A} = \begin{pmatrix} \sqrt{2}D\hat{f}(x) & 0 \\ 0 & \frac{1}{\sqrt{2}}\Delta_3(f_n) \end{pmatrix} \in \mathbb{C}^{n \times n},$$

since $D\hat{f}(x)$ is invertible and $\Delta_3(f_n) \neq 0$, we have

$$(3.5) \quad \mathcal{A}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}}D\hat{f}(x)^{-1} & 0 \\ 0 & \frac{\sqrt{2}}{\Delta_3(f_n)} \end{pmatrix}.$$

Lemma 2. *If $\gamma_3(f, x)\|w\| \leq \frac{1}{2}$, then*

$$\|\mathcal{A}^{-1}f(y)\| \geq \frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{1 + 2 \cos \varphi + \sin \varphi} \|w\|^3 - 2\gamma_3^3 \|w\|^4.$$

Proof. By Taylor's expansion of $\hat{f}(y)$ at x , and $\frac{\partial \hat{f}(x)}{\partial X_1} = 0$, we have

$$\eta = D\hat{f}(x)^{-1} \left(\hat{f}(y) - \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \zeta^2 - \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}} \zeta \eta - \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2} \eta^2 - \sum_{k \geq 3} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \right).$$

By expanding $f_n(y)$ at x and $\frac{\partial f_n(x)}{\partial X_1} = \dots = \frac{\partial f_n(x)}{\partial X_n} = \frac{\partial^2 f_n(x)}{\partial X_1^2} = 0$, we have:

$$\begin{aligned} f_n(y) &= \left(\frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \zeta \eta + \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \eta^2 \right) + \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1^3} \zeta^3 + \frac{1}{2} \frac{\partial^3 f_n(x)}{\partial X_1^2 \partial \hat{X}} \zeta^2 \eta \\ &\quad + \frac{1}{2} \frac{\partial^3 f_n(x)}{\partial X_1 \partial \hat{X}^2} \zeta \eta^2 + \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial \hat{X}^3} \eta^3 + \sum_{k \geq 4} \frac{D^k f_n(x)(y-x)^k}{k!}. \end{aligned}$$

Substituting one η in $\frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \zeta \eta$ and $\frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \eta^2$ by the expansion of η , as $\Delta_3(f_n) \neq 0$, we have

$$\begin{aligned} \frac{1}{\Delta_3(f_n)} f_n(y) &= \frac{1}{\Delta_3(f_n)} \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} D\hat{f}(x)^{-1} \hat{f}(y) \zeta + \frac{1}{\Delta_3(f_n)} \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} D\hat{f}(x)^{-1} \hat{f}(y) \eta + \zeta^3 \\ &+ \frac{1}{\Delta_3(f_n)} C_{2,1} \zeta^2 \eta + \frac{1}{\Delta_3(f_n)} C_{1,2} \zeta \eta^2 + \frac{1}{\Delta_3(f_n)} C_{0,3} \eta^3 + \sum_{k \geq 4} \frac{1}{\Delta_3(f_n)} \frac{D^k f_n(x)(y-x)^k}{k!} \\ &+ \frac{1}{\Delta_3(f_n)} T_{1,0} \sum_{k \geq 3} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta + \frac{1}{\Delta_3(f_n)} T_{0,1} \sum_{k \geq 3} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \eta. \end{aligned}$$

where

$$\begin{aligned} C_{2,1} &= \frac{1}{2} \frac{\partial^3 f_n(x)}{\partial X_1^2 \partial \hat{X}} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(x)^{-1} \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \cdot D\hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2}, \\ C_{1,2} &= \frac{1}{2} \frac{\partial^3 f_n(x)}{\partial X_1 \partial \hat{X}^2} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \cdot D\hat{f}(x)^{-1} \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}}, \\ C_{0,3} &= \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial \hat{X}^3} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \cdot D\hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2}, \\ T_{1,0} &= -\frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}}, \\ T_{0,1} &= -\frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2}. \end{aligned}$$

For the classical operator norm, we have the following inequalities for $i + j = k$:

$$\left\| \frac{\partial^k \hat{f}(x)}{\partial X_1^i \partial \hat{X}^j} \right\| \leq \|D^k \hat{f}(x)\|, \quad \left\| \frac{\partial^k f_n(x)}{\partial X_1^i \partial \hat{X}^j} \right\| \leq \|D^k f_n(x)\|.$$

Therefore, by moving ζ^3 to the left side and $\frac{1}{\Delta_3(f_n)} f_n(y)$ to the right side of the equation and applying the triangle inequalities, we have

$$\begin{aligned} |\zeta|^3 &\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + 2\gamma_{3,n} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| |\zeta| + \gamma_{3,n} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \|\eta\| \\ &+ (3\gamma_{3,n}^2 + 2\gamma_{3,n} \cdot 2\hat{\gamma}_3 + \gamma_{3,n} \hat{\gamma}_3) |\zeta|^2 \|\eta\| \\ &+ (3\gamma_{3,n}^2 + 2\gamma_{3,n} \hat{\gamma}_3 + \gamma_{3,n} \cdot 2\hat{\gamma}_3) |\zeta| \|\eta\|^2 + (\gamma_{3,n}^2 + \gamma_{3,n} \hat{\gamma}_3) \|\eta\|^3 \\ &+ \sum_{k \geq 4} \gamma_{3,n}^{k-1} \|w\|^k + 2\gamma_{3,n} \sum_{k \geq 3} \hat{\gamma}_3^{k-1} \|w\|^k |\zeta| + \gamma_{3,n} \sum_{k \geq 3} \hat{\gamma}_3^{k-1} \|w\|^k \|\eta\| \\ &\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| (2\gamma_{3,n} |\zeta| + \gamma_{3,n} \|\eta\|) \\ &+ (3\gamma_{3,n}^2 + 5\gamma_{3,n} \hat{\gamma}_3) |\zeta|^2 \|\eta\| + (3\gamma_{3,n}^2 + 4\gamma_{3,n} \hat{\gamma}_3) |\zeta| \|\eta\|^2 \\ &+ (\gamma_{3,n}^2 + \gamma_{3,n} \hat{\gamma}_3) \|\eta\|^3 + 2\gamma_{3,n}^3 \|w\|^4 + 4\gamma_{3,n} \hat{\gamma}_3^2 \|w\|^3 |\zeta| + 2\gamma_{3,n} \hat{\gamma}_3^2 \|w\|^3 \|\eta\| \\ &\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| (2\gamma_{3,n} |\zeta| + \gamma_{3,n} \|\eta\|) + 8\gamma_{3,n}^2 |\zeta|^2 \|\eta\| \\ &+ 7\gamma_{3,n}^2 |\zeta| \|\eta\|^2 + 2\gamma_{3,n}^2 \|\eta\|^3 + 2\gamma_{3,n}^3 \|w\|^4 + 4\gamma_{3,n}^3 \|w\|^3 |\zeta| + 2\gamma_{3,n}^3 \|w\|^3 \|\eta\|, \end{aligned}$$

where the second inequality follows from the assumption that $\hat{\gamma}_3(f, x)\|w\| \leq \frac{1}{2}$ and $\gamma_{3,n}(f, x)\|w\| \leq \frac{1}{2}$, while the last inequality attributes to the fact that $\gamma_3 = \max(\hat{\gamma}_3, \gamma_{3,n})$.

As $|\zeta| = \|w\| \cos \varphi$, $\|\eta\| = \|w\| \sin \varphi$, we have

$$\begin{aligned} \|w\|^3 \cos^3 \varphi &\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \gamma_{3,n} \|w\| (2 \cos \varphi + \sin \varphi) \\ &\quad + 2\gamma_3^2 \|w\|^3 \sin^3 \varphi + 7\gamma_3^2 \|w\|^3 \cos \varphi \sin^2 \varphi + 8\gamma_3^2 \|w\|^3 \cos^2 \varphi \sin \varphi \\ &\quad + 2\gamma_3^3 \|w\|^4 (1 + 2 \cos \varphi + \sin \varphi) \end{aligned}$$

For $\varphi \in [0, \frac{\pi}{2}]$, $1 \leq 2 \cos \varphi + \sin \varphi \leq \sqrt{5}$, we have

$$\begin{aligned} &\frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{1 + 2 \cos \varphi + \sin \varphi} \|w\|^3 - 2\gamma_3^3 \|w\|^4 \\ &\leq \frac{1}{1 + 2 \cos \varphi + \sin \varphi} \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \frac{\gamma_{3,n} w (2 \cos \varphi + \sin \varphi)}{1 + 2 \cos \varphi + \sin \varphi} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \\ &\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \frac{\sqrt{5}}{2 + 2\sqrt{5}} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \\ &\leq \sqrt{2} \left\| \begin{pmatrix} \frac{\sqrt{5}}{2+2\sqrt{5}} D\hat{f}(x)^{-1} \hat{f}(y) \\ \frac{1}{\Delta_3(f_n)} f_n(y) \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} \frac{1}{\sqrt{2}} D\hat{f}(x)^{-1} & 0 \\ 0 & \frac{\sqrt{2}}{\Delta_3(f_n)} \end{pmatrix} \begin{pmatrix} \hat{f}(y) \\ f_n(y) \end{pmatrix} \right\| \\ &= \|\mathcal{A}^{-1} f(y)\|. \end{aligned}$$

□

Lemma 3. Let $d \approx 0.08507$ be the positive root of the equation

$$(3.6) \quad (1 - 2d - 8d^2) \sqrt{1 - d^2} - 9d - d^2 + 6d^3 = 0.$$

Let θ be defined by

$$(3.7) \quad \sin \theta = \frac{d}{\gamma_3^2}.$$

Then, for $\gamma_3(f, x)\|w\| \leq \frac{1}{2}$ and $\forall y \in \mathbb{C}^n$, either

$$\theta \leq \varphi \leq \frac{\pi}{2} \text{ and } \|\mathcal{A}^{-1} f(y)\| \geq \sqrt{2} \gamma_3 \|w\| \left(\frac{\sin \theta}{2\gamma_3} - \|w\| \right),$$

or

$$0 \leq \varphi \leq \theta \text{ and } \|\mathcal{A}^{-1} f(y)\| \geq 2\gamma_3^3 \|w\|^3 \left(\frac{\sin \theta}{2\gamma_3} - \|w\| \right).$$

Proof. For $\theta \leq \varphi \leq \frac{\pi}{2}$, by Lemma 1, we have

$$\begin{aligned} \sqrt{2} \|\mathcal{A}^{-1} f(y)\| &= \left\| \begin{pmatrix} D\hat{f}(x)^{-1} \hat{f}(y) \\ \frac{2}{\Delta_3(f_n)} f_n(y) \end{pmatrix} \right\| \geq \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \\ &\geq \|w\| \sin \theta - 2\hat{\gamma}_3(f, x)\|w\|^2 \geq 2\gamma_3 \|w\| \left(\frac{\sin \theta}{2\gamma_3} - \|w\| \right). \end{aligned}$$

For $0 \leq \varphi \leq \theta$, by Lemma 2, we have

$$\|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_3^3\|w\|^3 \left(\frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{2\gamma_3^3(1 + 2 \cos \varphi + \sin \varphi)} - \|w\| \right).$$

Let

$$h(\varphi) = \frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{2\gamma_3^3(1 + 2 \cos \varphi + \sin \varphi)}.$$

We claim that

$$h(\theta) \geq \frac{\sin \theta}{2\gamma_3}.$$

To prove this claim, as $\sin \theta = \frac{d}{\gamma_3}$, it is sufficient to show that

$$\left(1 - \frac{d^2}{\gamma_3^4}\right)^{\frac{3}{2}} - 8d \left(1 - \frac{d^2}{\gamma_3^4}\right) - \frac{7d^2}{\gamma_3^2} \sqrt{1 - \frac{d^2}{\gamma_3^4}} - \frac{2d^3}{\gamma_3^4} - d - 2d \sqrt{1 - \frac{d^2}{\gamma_3^4}} - \frac{d^2}{\gamma_3^2} \geq 0.$$

Since this function for $\gamma_3 \geq 1$, is increasing for any $d \in [0, \frac{1}{6}]$, similar to the proof of [8, Lemma 4], it is sufficient to check this inequality for $\gamma_3 = 1$,

$$(1 - 2d - 8d^2)\sqrt{1 - d^2} - 9d - d^2 + 6d^3 \geq 0.$$

The smallest positive root of the equation (3.6) obtained by setting the above inequality to 0 is

$$d \approx 0.08507,$$

which lies in $[0, \frac{1}{6}]$. The claim $h(\theta) \geq \frac{\sin \theta}{2\gamma_3}$ follows.

Furthermore, the polynomial $h(\varphi)$ is non-negative and decreasing for

$$(3.8) \quad \varphi \in [0, \theta], \quad \theta \in \left[0, \arcsin \frac{2}{\sqrt{5}}\right],$$

as its numerator is decreasing and its denominator is increasing, and both are non-negative for φ satisfying (3.8). Hence, we have

$$h(\varphi) \geq h(\theta) \geq \frac{\sin \theta}{2\gamma_3},$$

for φ satisfying (3.8). Together with Lemma 2, we have

$$\|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_3^3\|w\|^3 (h(\varphi) - \|w\|) \geq 2\gamma_3^3\|w\|^3 \left(\frac{\sin \theta}{2\gamma_3} - \|w\| \right).$$

□

Let $d \approx 0.08507$ be the smallest positive root of the equation (3.6). The following four theorems generalize the results in [8] to simple triple zeros.

Theorem 1. *Let x be an isolated simple triple zero of the polynomial system f , and y is another zero of f , then*

$$(3.9) \quad \|y - x\| \geq \frac{d}{2\gamma_3^3}.$$

Proof. Since $f(y) = 0$, when $\gamma_3\|w\| \leq \frac{1}{2}$, by Lemma 3 and (3.7), we have

$$\|y - x\| = \|w\| \geq \frac{\sin \theta}{2\gamma_3} = \frac{d}{2\gamma_3^3}.$$

For $\gamma_3\|w\| \geq \frac{1}{2}$, the same conclusion holds as $\gamma_3 \geq 1$ and $d < 1$.

□

Theorem 2. *Let x be an isolated simple triple zero of the polynomial system f , and $\|y - x\| \leq \frac{d}{4\gamma_3^3}$, then*

$$\|f(y)\| \geq \frac{d\|y - x\|^3}{2\|\mathcal{A}^{-1}\|}.$$

Proof. When

$$\|w\| = \|y - x\| \leq \frac{d}{4\gamma_3^3} = \frac{\sin \theta}{4\gamma_3},$$

by Lemma 3, we have

$$\|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_3^3\|w\|^3 \left(\frac{\sin \theta}{2\gamma_3} - \|w\| \right) \geq 2\gamma_3^3\|w\|^3 \frac{\sin \theta}{4\gamma_3} = 2\gamma_3^3\|w\|^3 \frac{d}{4\gamma_3^3} = \frac{d}{2}\|w\|^3.$$

□

For $R > 0$, let us define

$$(3.10) \quad d_R(f, g) = \max_{\|y-x\| \leq R} \|f(y) - g(y)\|.$$

Theorem 3. *Let x be an isolated simple triple zero of the polynomial system f and*

$$0 < R \leq \frac{d}{4\gamma_3^3}.$$

If

$$d_R(f, g) < \frac{dR^3}{2\|\mathcal{A}^{-1}\|},$$

then the sum of the multiplicities of the zeros of g in $B(x, R)$ is three.

Proof. By Theorem 2, for any y such that $\|y - x\| = R$, we have

$$\|f(y) - g(y)\| \leq d_R(f, g) < \frac{dR^3}{2\|\mathcal{A}^{-1}\|} = \frac{d\|y - x\|^3}{2\|\mathcal{A}^{-1}\|} \leq \|f(y)\|,$$

by Rouché's Theorem, f and g have the same number of zeros inside $B(x, R)$. By Theorem 1, when $R \leq \frac{d}{4\gamma_3^3}$, the only zero of f in $B(x, R)$ is x . Therefore, g has three zeros in $B(x, R)$. □

Given $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $x \in \mathbb{C}^n$, such that $D\hat{f}(x)$ is invertible, and

$$\Delta_3(f_n) = \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1^3} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} D\hat{f}(x)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \neq 0.$$

We define tensors

$$H_1 = \begin{pmatrix} \frac{\partial \hat{f}(x)}{\partial X_1} & 0 \\ \frac{\partial f_n(x)}{\partial X_1} & \frac{\partial f_n(x)}{\partial \hat{X}} \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} & 0 \end{pmatrix} \mathbf{0}_{n \times n \times (n-1)} \end{pmatrix},$$

and polynomials

$$g(X) = f(X) - f(x) - H_1(X - x) - H_2(X - x)^2.$$

Theorem 4. Let $\gamma_3 = \gamma_3(g, x)$, if

$$\|f(x)\| + \|H_1\| \frac{d}{4\gamma_3^3} + \|H_2\| \frac{d^2}{16\gamma_3^6} < \frac{d^4}{128\gamma_3^9 \|\mathcal{A}^{-1}\|},$$

then f has three zeros (counting multiplicities) in the ball of radius $\frac{d}{4\gamma_3^3}$ around x .

Proof. We have $g(x) = 0$,

$$Dg(x) = Df(x) - H_1 = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix},$$

Moreover, we have

$$\begin{aligned} \Delta_2(g_n) &= \frac{1}{2} \frac{\partial^2 g_n(x)}{\partial X_1^2} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} = 0, \\ \Delta_3(g_n) &= \frac{1}{6} \frac{\partial^3 g_n(x)}{\partial X_1^3} - \frac{\partial^2 g_n(x)}{\partial X_1 \partial \hat{X}} \cdot D\hat{g}(x)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{g}(x)}{\partial X_1^2} \\ &= \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1^3} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(x)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \neq 0. \end{aligned}$$

Hence $Dg(x)$ satisfies the normalized form, and x is a simple singular root of g with multiplicity three. Let $R = \frac{d}{4\gamma_3^3}$, we have

$$\begin{aligned} d_R(g, f) &= \max_{\|y-x\| \leq R} \|g(y) - f(y)\| \\ &= \max_{\|y-x\| \leq R} \|f(x) + H_1(y-x) + H_2(y-x)^2\| \\ &\leq \|f(x)\| + \|H_1\|R + \|H_2\|R^2 \\ &= \|f(x)\| + \|H_1\| \frac{d}{4\gamma_3^3} + \|H_2\| \frac{d^2}{16\gamma_3^6}. \end{aligned}$$

If

$$\|f(x)\| + \|H_1\| \frac{d}{4\gamma_3^3} + \|H_2\| \frac{d^2}{16\gamma_3^6} < \frac{d^4}{128\gamma_3^9 \|\mathcal{A}^{-1}\|},$$

then

$$d_R(g, f) < \frac{d^4}{128\gamma_3^9 \|\mathcal{A}^{-1}\|} = \frac{dR^3}{2\|\mathcal{A}^{-1}\|}.$$

By Theorem 3, the sum of the multiplicities of the zeros of f in $B(x, R)$ is three. \square

Remark 2. The equality of $\gamma_\mu(g, x) = \gamma_\mu(f, x)$ is true for $\mu = 2$ [8, Theorem 4]. In Example 2, we show that $\left\| \frac{1}{\Delta_3(f_2)} \cdot \frac{D^2 f_2(x)}{2} \right\| \neq \left\| \frac{1}{\Delta_3(g_2)} \cdot \frac{D^2 g_2(x)}{2} \right\|$. Hence, $\gamma_{3,n}(g, x)$ might be not equal to $\gamma_{3,n}(f, x)$ if they are not equal to 1.

3.2. Simple Multiple Zeros. We generalize results in Section 3.1 to the simple multiple zeros of higher multiplicities.

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, and x be a simple zero of f of multiplicity μ , where $Df(x)$ has the normalized form $Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix}$, $D\hat{f}(x)$ is invertible and

$$(3.11) \quad \Delta_k(f_n) = 0, \text{ for } k = 2, \dots, \mu - 1, \quad \Delta_\mu(f_n) \neq 0.$$

Let y be another vector in \mathbb{C}^n and $y \neq x$. Recall that $\varphi = d_P(v, y - x)$, $v = (1, 0, \dots, 0)^T$ and $w = x - y = (\zeta, \eta_2, \dots, \eta_n)^T$, $\eta = (\eta_2, \dots, \eta_n)^T$, then we have $|\zeta| = \|w\| \sin \varphi$, $\|\eta\| = \|w\| \cos \varphi$. Let

$$\mathcal{A} = \begin{pmatrix} \sqrt{2} D\hat{f}(x) & 0 \\ 0 & \frac{1}{\sqrt{2}} \Delta_\mu(f_n) \end{pmatrix},$$

and $\gamma_\mu = \max(\hat{\gamma}_\mu, \gamma_{\mu,n})$, where

$$\gamma_{\mu,n} = \gamma_{\mu,n}(f, x) = \max \left(1, \sup_{k \geq 2} \left\| \frac{1}{\Delta_\mu(f_n)} \cdot \frac{D^k f_n(x)}{k!} \right\|^{\frac{1}{k-1}} \right).$$

Case 1: For $\theta \leq \varphi \leq \frac{\pi}{2}$, assume that $\gamma_\mu \|w\| \leq \frac{1}{2}$. The Taylor's expansion of $\hat{f}(y)$ at x is:

$$\hat{f}(y) = D\hat{f}(x)\eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \zeta^2 + \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}} \zeta \eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2} \eta^2 + \sum_{k \geq 3} \frac{D^k \hat{f}(x)(y-x)^k}{k!}.$$

By the triangle inequality, we have

$$\begin{aligned} \|w\| \sin \varphi = \|\eta\| &\leq \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| + \sum_{k \geq 2} \hat{\gamma}_\mu(f, x)^{k-1} \|w\|^k \\ &\leq \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| + 2\hat{\gamma}_\mu(f, x) \|w\|^2. \end{aligned}$$

Therefore, we have the following claim.

Claim 1. For $\theta \leq \varphi \leq \frac{\pi}{2}$, assume that $\gamma_\mu \|w\| \leq \frac{1}{2}$, we have

$$\left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \geq 2\gamma_\mu \|w\| \left(\frac{\sin \theta}{2\gamma_\mu} - \|w\| \right),$$

and

$$\|w\| \geq \frac{\sin \varphi}{2\hat{\gamma}_\mu} \geq \frac{\sin \theta}{2\hat{\gamma}_\mu} \geq \frac{\sin \theta}{2\gamma_\mu}.$$

Case 2: For $0 \leq \varphi < \theta \leq \frac{\pi}{2}$, assume that $\gamma_\mu \|w\| \leq \frac{1}{2}$. The Taylor expansion of f_n at y is:

$$\begin{aligned} f_n(y) &= \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} \zeta^2 + \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \zeta \eta + \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \eta^2 + \dots + \frac{1}{\mu!} \frac{\partial^\mu f_n(x)}{\partial X_1^\mu} \zeta^\mu \\ &\quad + \frac{1}{(\mu-1)!} \frac{\partial^\mu f_n(x)}{\partial X_1^{\mu-1} \partial \hat{X}} \zeta^{\mu-1} \eta + \dots + \frac{1}{\mu!} \frac{\partial^\mu f_n(x)}{\partial \hat{X}^\mu} \eta^\mu + \sum_{k \geq \mu+1} \frac{D^k f_n(x)(y-x)^k}{k!}. \end{aligned}$$

The coefficient of the term $\zeta^i \eta^j$ in the Taylor expansion of f_n is $\frac{1}{i!j!} \frac{\partial^{i+j} f_n(x)}{\partial X_1^i \partial \hat{X}^j}$, whose norm divided by $\Delta_\mu(f_n)$ is bounded by

$$(3.12) \quad \frac{(i+j)!}{i!j!} \gamma_\mu^{i+j-1}.$$

For the monomial $\zeta^i \eta^j$, $i+j < \mu$ and $j > 0$, after substituting the first η in $\zeta^i \eta^j$ by

$$\eta = -D\hat{f}(x)^{-1} \left(\frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \zeta^2 + \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}} \zeta \eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2} \eta^2 + \dots \right)$$

$$\begin{aligned}
& + \sum_{0 \leq k \leq \mu+1-i-j} \frac{1}{(\mu+1-i-j-k)!k!} \frac{\partial^{\mu+1-i-j} \hat{f}(x)}{\partial X_1^{\mu+1-i-j-k} \partial \hat{X}^k} \zeta^{\mu+1-i-j-k} \eta^k \\
& + \sum_{k \geq \mu+2-i-j} \left(\frac{D^k \hat{f}(x)(y-x)^k}{k!} - \hat{f}(y) \right),
\end{aligned}$$

solved from the Taylor's expansion formula for $\hat{f}(y)$ at x , we have

$$\begin{aligned}
\zeta^i \eta^j &= -D\hat{f}(x)^{-1} \left(\frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \zeta^{i+2} \eta^{j-1} + \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial \hat{X}} \zeta^{i+1} \eta^j + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial \hat{X}^2} \zeta^i \eta^{j+1} + \dots \right. \\
& + \sum_{0 \leq k \leq \mu+1-i-j} \frac{1}{(\mu+1-i-j-k)!k!} \frac{\partial^{\mu+1-i-j} \hat{f}(x)}{\partial X_1^{\mu+1-i-j-k} \partial \hat{X}^k} \zeta^{\mu+1-i-j-k} \eta^{k+j-1} \\
& \left. + \sum_{k+i+j-1 \geq \mu+1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^{j-1} - \hat{f}(y) \zeta^i \eta^{j-1} \right),
\end{aligned}$$

the total degree of each term in the above expression is at least $i+j+1$. Moreover, the norm of the coefficient of the new term $\zeta^{i+k} \eta^{j-1+l}$, $i+k+j-1+l \leq \mu$ obtained after the substitution and divided by $\Delta_\mu(f_n)$ is bounded by

$$(3.13) \quad \left(\frac{(i+j)!}{i!j!} \gamma_\mu^{i+j-1} \right) \left\| D\hat{f}(x)^{-1} \frac{1}{k!l!} \frac{\partial^{k+l} \hat{f}(x)}{\partial X_1^k \partial \hat{X}^l} \right\| \leq \frac{(i+j)!}{i!j!} \frac{(k+l)!}{k!l!} \gamma_\mu^{i+k+j+l-2}.$$

Starting from $i+j=2, j \geq 1$, after at most $\mu-2$ substitutions, we can write f_n in the following form:

(3.14)

$$\begin{aligned}
f_n(y) &= C_2 \zeta^2 + \dots + C_\mu \zeta^\mu + \sum_{i+j=\mu, j>0} C_{i,j} \zeta^i \eta^j + \sum_{k \geq \mu+1} \frac{D^k f_n(x)(y-x)^k}{k!} \\
& + \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} \cdot \left(\sum_{k+i+j-1 \geq \mu+1} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^{j-1} \right) \\
& - \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} D\hat{f}(x)^{-1} \hat{f}(y) \zeta^i \eta^{j-1},
\end{aligned}$$

where C_2, \dots, C_μ are constants, and the coefficients $C_{i,j}$ and $T_{i,j-1}$ divided by $\Delta_\mu(f_n)$ are bounded:

$$(3.15) \quad \left\| \frac{1}{\Delta_\mu(f_n)} C_{i,j} \right\| \leq c_{i,j} \gamma_\mu^{i+j-1}, \quad \left\| \frac{1}{\Delta_\mu(f_n)} T_{i,j-1} \right\| \leq t_{i,j-1} \gamma_\mu^{i+j-1},$$

where $c_{i,j}, t_{i,j-1} \in \mathbb{R}$ are constants. This can be deduced by using (3.12) and (3.13).

Claim 2. We have $C_t = \Delta_t(f_n)$, for $t = 2, \dots, \mu$.

For simplicity, we replace $j-1$ by j in the last two terms of (3.14).

Proof. Let us apply the differential functional Δ_t to both sides of (3.14):

$$\Delta_t(f_n) = C_2 \Delta_t(\zeta^2) + \dots + C_\mu \Delta_t(\zeta^\mu) + \sum_{i+j=\mu, j>0} C_{i,j} \Delta_t(\zeta^i \eta^j)$$

$$\begin{aligned}
& + \sum_{1 \leq i+j \leq \mu-2} T_{i,j} \cdot \left(\sum_{k \geq \mu+1-i-j} D\hat{f}(x)^{-1} \Delta_t \left(\frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^j \right) \right) \\
& - \sum_{1 \leq i+j \leq \mu-2} T_{i,j} D\hat{f}(x)^{-1} \Delta_t \left(\hat{f}(y) \zeta^i \eta^j \right) + \sum_{k \geq \mu+1} \Delta_t \left(\frac{D^k f_n(x)(y-x)^k}{k!} \right).
\end{aligned}$$

Based on (2.2), (2.4) and the fact that d_1^t is the only differential monomial of the highest order t in Δ_t and no other d_1^s with $s < t$ in Δ_t , we derive for $2 \leq t \leq \mu$ that:

- (1) $\Delta_t(\zeta^s) = 1$ if $s = t$ and 0 otherwise;
- (2) $\Delta_t(\zeta^i \eta^j) = 0$ for $t \leq i+j = \mu$ and $j > 0$;
- (3) $\Delta_t \left(\frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^j \right) = 0$ for $t \leq \mu < i+j+k$;
- (4) $\Delta_t \left(\hat{f}(y) \zeta^i \eta^j \right) = 0$ for $1 \leq i+j \leq \mu-2$;
- (5) $\Delta_t \left(\frac{D^k f_n(x)(y-x)^k}{k!} \right) = 0$ for $t \leq \mu < k$.

Hence, we have $C_t = \Delta_t(f_n)$ for $t = 2, \dots, \mu$. □

By Claim 2 and (3.11), we have

$$(3.16) \quad C_t = \Delta_t(f_n) = 0, \quad t = 2, \dots, \mu-1, \quad C_\mu = \Delta_\mu(f_n) \neq 0.$$

From (3.14) and (3.16), we obtain

$$\begin{aligned}
\zeta^\mu = & - \frac{1}{\Delta_\mu(f_n)} \sum_{i+j=\mu, j>0} C_{i,j} \zeta^i \eta^j - \frac{1}{\Delta_\mu(f_n)} \sum_{k \geq \mu+1} \frac{D^k f_n(x)(y-x)^k}{k!} \\
& - \frac{1}{\Delta_\mu(f_n)} \sum_{1 \leq i+j \leq \mu-2} T_{i,j} \cdot \left(\sum_{k \geq \mu+1-i-j} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^j \right) \\
& + \frac{1}{\Delta_\mu(f_n)} \sum_{1 \leq i+j \leq \mu-2} T_{i,j} D\hat{f}(x)^{-1} \hat{f}(y) \zeta^i \eta^j + \frac{1}{\Delta_\mu(f_n)} f_n(y).
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}
|\zeta|^\mu \leq & \sum_{i+j=\mu, j>0} \left\| \frac{1}{\Delta_\mu(f_n)} C_{i,j} \right\| |\zeta|^i \|\eta\|^j + \sum_{k \geq \mu+1} \left\| \frac{1}{\Delta_\mu(f_n)} \frac{D^k f_n(x)}{k!} \right\| \|w\|^k \\
& + \sum_{1 \leq i+j \leq \mu-2} \left\| \frac{1}{\Delta_\mu(f_n)} T_{i,j} \right\| \cdot \left(\sum_{k \geq \mu+1-i-j} \left\| D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!} \right\| \|w\|^k |\zeta|^i \|\eta\|^j \right) \\
& + \sum_{1 \leq i+j \leq \mu-2} \left\| \frac{1}{\Delta_\mu(f_n)} T_{i,j} \right\| \cdot \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| |\zeta|^i \|\eta\|^j + \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right| \\
\leq & \sum_{i+j=\mu, j>0} c_{i,j} \gamma_\mu^{i+j-1} |\zeta|^i \|\eta\|^j + \sum_{k \geq \mu+1} \gamma_{\mu,n}^{k-1} \|w\|^k \\
& + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_\mu^{i+j} \cdot 2 \gamma_\mu^{\mu-i-j} \|w\|^{\mu-i-j+1} |\zeta|^i \|\eta\|^j \\
& + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_\mu^{i+j} \cdot \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| |\zeta|^i \|\eta\|^j + \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i+j=\mu, j>0} c_{i,j} \gamma_\mu^{\mu-1} |\zeta|^i \|\eta\|^j + \sum_{1 \leq i+j \leq \mu-2} 2t_{i,j} \gamma_\mu^\mu \|w\|^{\mu-i-j+1} |\zeta|^i \|\eta\|^j + 2\gamma_\mu^\mu \|w\|^{\mu+1} \\
&\quad + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_\mu^{i+j} \cdot \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| |\zeta|^i \|\eta\|^j + \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right|.
\end{aligned}$$

By $|\zeta| = \|w\| \sin \varphi$, $\|\eta\| = \|w\| \cos \varphi$, we have:

$$\begin{aligned}
&\|w\|^\mu \cos^\mu \varphi \leq \sum_{i+j=\mu, j>0} c_{i,j} \gamma_\mu^{\mu-1} \|w\|^\mu \cos^i \varphi \sin^j \varphi \\
&\quad + \sum_{1 \leq i+j \leq \mu-2} 2t_{i,j} \gamma_\mu^\mu \|w\|^{\mu+1} \cos^i \varphi \sin^j \varphi + 2\gamma_\mu^\mu \|w\|^{\mu+1} \\
&\quad + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_\mu^{i+j} \|w\|^{i+j} \cos^i \varphi \sin^j \varphi \cdot \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| + \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\cos^\mu \varphi - \sum_{i+j=\mu, j>0} c_{i,j} \gamma_\mu^{\mu-1} \cos^i \varphi \sin^j \varphi}{1 + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \cos^i \varphi \sin^j \varphi} \cdot \|w\|^\mu - 2\gamma_\mu^\mu \|w\|^{\mu+1} \\
&\leq \frac{1}{1 + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \cos^i \varphi \sin^j \varphi} \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right| \\
&\quad + \frac{\sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_\mu^{i+j} \|w\|^{i+j} \cos^i \varphi \sin^j \varphi}{1 + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \cos^i \varphi \sin^j \varphi} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \\
&\leq \left| \frac{1}{\Delta_\mu(f_n)} f_n(y) \right| + \frac{1}{2} \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \\
&\leq \|\mathcal{A}^{-1} f(y)\|.
\end{aligned}$$

We have the following inequality:

$$(3.17) \quad \|\mathcal{A}^{-1} f(y)\| \geq h(\varphi) \cdot 2\gamma_\mu^\mu \|w\|^\mu - 2\gamma_\mu^\mu \|w\|^{\mu+1}.$$

where

$$(3.18) \quad h(\varphi) = \frac{\cos^\mu \varphi - \sum_{i+j=\mu, j>0} c_{i,j} \gamma_\mu^{\mu-1} \cos^i \varphi \sin^j \varphi}{\sum_{1 \leq i \leq \mu-2} 2t_{i,0} \gamma_\mu^\mu + \sum_{1 \leq i+j \leq \mu-2, j>0} 2t_{i,j} \gamma_\mu^\mu \cos^i \varphi \sin^j \varphi + 2\gamma_\mu^\mu}.$$

Definition 3. We define $d = \min(d_1, d_2, d_3)$, where

$$d_1 = \sqrt{\frac{1}{c_{\mu-1,1}^2 + 1}}, \quad d_2 = \sqrt{\frac{1}{\mu-1}},$$

and d_3 is the smallest positive real root of the polynomial

$$\begin{aligned}
(3.19) \quad &p(d) = (1-d^2)^{\frac{\mu}{2}} - \sum_{i+j=\mu, j>0} c_{i,j} d(1-d^2)^{\frac{i}{2}} d^{j-1} \\
&\quad - d \left(\sum_{1 \leq i \leq \mu-2} t_{i,0} + \sum_{1 \leq i+j \leq \mu-2, j>0} t_{i,j} (1-d^2)^{\frac{i}{2}} d^j + 1 \right).
\end{aligned}$$

In the sequel, we always assume that d has the above definition.

Claim 3. We have $h(\theta) \geq \frac{\sin(\theta)}{2\gamma_\mu}$, where $\sin \theta = \frac{d}{\gamma_\mu^{\mu-1}}$.

To prove this claim, substituting $\sin \varphi$ and $\cos \varphi$ in (3.18) by $\sin \theta = \frac{d}{\gamma_\mu^{\mu-1}}$, $\cos \theta = \left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{1/2}$, we need to show

$$(3.20) \quad \begin{aligned} & \left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{\mu}{2}} - c_{\mu-1,1}d \left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{\mu-1}{2}} \\ & - \sum_{i+j=\mu-1, j>0} c_{i,j+1}d \left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{i}{2}} \frac{d^j}{\gamma_\mu^{j(\mu-1)}} \\ & - d \left(\sum_{1 \leq i \leq \mu-2} t_{i,0} + \sum_{1 \leq i+j \leq \mu-2, j>0} t_{i,j} \left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{i}{2}} \frac{d^j}{\gamma_\mu^{j(\mu-1)}} + 1 \right) \geq 0. \end{aligned}$$

- The sum of the first two terms in (3.20) is non-negative and increasing in γ_μ for $\gamma_\mu \geq 1$ as it equals to

$$\left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{\mu-1}{2}} \left(\sqrt{1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}} - c_{\mu-1,1}d \right),$$

and $d \leq d_1 = \sqrt{\frac{1}{c_{\mu-1,1}^2 + 1}}$.

- The terms $\cos^i \varphi \sin^j \varphi$, $j > 0$ are increasing with respect to φ for $\varphi \in \left[0, \arctan \sqrt{\frac{1}{i}}\right]$ since

$$(\cos^i \varphi \sin \varphi)' = \cos^{i-1} \varphi (\cos^2 \varphi - i \sin^2 \varphi) \geq 0.$$

Hence, for $1 \leq i+j \leq \mu-1$, $j > 0$, $\left(1 - \frac{d^2}{\gamma_\mu^{2(\mu-1)}}\right)^{\frac{i}{2}} \frac{d^j}{\gamma_\mu^{j(\mu-1)}}$, is decreasing in γ_μ for $d \in \left[0, \sqrt{\frac{1}{\mu-1}}\right]$.

- The left side of (3.20) is a function which is increasing in γ_μ and it is sufficient to prove that the inequality is true when $\gamma_\mu = 1$:

$$\begin{aligned} p(d) &= (1 - d^2)^{\frac{\mu}{2}} - \sum_{i+j=\mu, j>0} c_{i,j}d(1 - d^2)^{\frac{i}{2}} d^{j-1} \\ & - d \left(\sum_{1 \leq i \leq \mu-2} t_{i,0} + \sum_{1 \leq i+j \leq \mu-2, j>0} t_{i,j}(1 - d^2)^{\frac{i}{2}} d^j + 1 \right) \geq 0, \end{aligned}$$

which is obvious as $p(d)$ is decreasing in d for $0 \leq d \leq d_3$, $p(0) > 0$ and d_3 is the smallest real zero of $p(d) = 0$.

Claim 4. For $0 \leq \varphi \leq \theta$, $h(\varphi)$ is non-negative and decreasing for $0 \leq \varphi \leq \theta$ and

$$(3.21) \quad h(\varphi) \geq h(\theta) \geq \frac{\sin \theta}{2\gamma_\mu},$$

and

$$(3.22) \quad \|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_\mu^\mu \|w\|^\mu \left(\frac{\sin \theta}{2\gamma_\mu} - \|w\| \right).$$

Moreover, if y is another zero of f , then

$$\|w\| = \|y - x\| \geq \frac{\sin \theta}{2\gamma_\mu}.$$

For $\varphi \in \left[0, \arctan \sqrt{\frac{1}{\mu-1}}\right]$, since $\cos^i \varphi \sin^j \varphi$, $i+j = \mu$, $j > 0$ is increasing, the numerator of $h(\varphi)$ is non-negative and decreasing, and the denominator of $h(\varphi)$ is positive and increasing. Hence, $h(\varphi)$ is non-negative and decreasing for $0 \leq \varphi \leq \theta$, we have (3.21). Moreover, by (3.17), we have

$$\|\mathcal{A}^{-1}f(y)\| \geq h(\varphi) \cdot 2\gamma_\mu^\mu \|w\|^\mu - 2\gamma_\mu^\mu \|w\|^{\mu+1} \geq 2\gamma_\mu^\mu \|w\|^\mu \left(\frac{\sin \theta}{2\gamma_\mu} - \|w\| \right).$$

Theorem 5. *Let x be a simple multiple zero of f of multiplicity μ , and y be another zero of f , then*

$$\|y - x\| \geq \frac{d}{2\gamma_\mu^\mu}.$$

Proof. By Claim 1 and Claim 4, we have

$$\|w\| = \|y - x\| \geq \frac{\sin \theta}{2\gamma_\mu} = \frac{d}{2\gamma_\mu^\mu},$$

since $\sin \theta = \frac{d}{\gamma_\mu^{\mu-1}}$. □

Theorem 6. *Let x be a simple multiple zero of f of multiplicity μ , and $\|y - x\| \leq \frac{d}{4\gamma_\mu^\mu}$, then we have*

$$\|f(y)\| \geq \frac{d\|y - x\|^\mu}{2\|\mathcal{A}^{-1}\|}.$$

Proof. For $\theta \leq \varphi \leq \frac{\pi}{2}$, by Claim 1, we can show that

$$\begin{aligned} \|\mathcal{A}^{-1}f(y)\| &= \left\| \begin{pmatrix} \frac{1}{\sqrt{2}} D\hat{f}(x)^{-1} \hat{f}(y) \\ \frac{\sqrt{2}}{\Delta_\mu(\hat{f}_n)} f_n(y) \end{pmatrix} \right\| \geq \frac{1}{\sqrt{2}} \|D\hat{f}(x)^{-1} \hat{f}(y)\| \\ &\geq \sqrt{2}\gamma_\mu \|w\| \left(\frac{\sin \theta}{2\gamma_\mu} - \|w\| \right). \end{aligned}$$

For $0 \leq \varphi \leq \theta$, by Claim 4, we have

$$\|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_\mu^\mu \|w\|^\mu \left(\frac{\sin \theta}{2\gamma_\mu} - \|w\| \right).$$

When $\|w\| = \|y - x\| \leq \frac{d}{4\gamma_\mu^\mu} = \frac{\sin \theta}{4\gamma_\mu}$, we have

$$\|\mathcal{A}^{-1}f(y)\| \geq 2\gamma_\mu^\mu \|w\|^\mu \frac{\sin \theta}{4\gamma_\mu} = \frac{d\|y - x\|^\mu}{2}.$$

□

Theorem 7. *Let x be a simple multiple zero of f of multiplicity μ and*

$$0 < R \leq \frac{d}{4\gamma_\mu^\mu}.$$

If

$$d_R(f, g) < \frac{dR^\mu}{2\|\mathcal{A}^{-1}\|},$$

then the sum of the multiplicities of the zeros of g in $B_R(x)$ is μ .

Proof. By Theorem 6, for any y such that $\|y - x\| = R < \frac{d}{4\gamma_\mu^\mu}$, we have

$$\|f(y) - g(y)\| \leq d_R(f, g) < \frac{dR^\mu}{2\|\mathcal{A}^{-1}\|} = \frac{d\|y - x\|^\mu}{2\|\mathcal{A}^{-1}\|} \leq \|f(y)\|,$$

by Rouché's Theorem, f and g have the same number of zeros inside $B_R(x)$. By Theorem 5, when $R \leq \frac{d}{4\gamma_\mu^\mu}$, the only root of f in $B_R(x)$ is x . Therefore, g has μ zeros in $B_R(x)$. \square

Given $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $x \in \mathbb{C}^n$, such that $D\hat{f}(x)$ is invertible, and $\Delta_\mu(f_n) \neq 0$, we define tensors

$$H_1 = \begin{pmatrix} \frac{\partial \hat{f}(x)}{\partial X_1} & 0 \\ \frac{\partial f_n(x)}{\partial X_1} & \frac{\partial f_n(x)}{\partial X} \end{pmatrix}$$

$$H_k = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ \Delta_k(f_n) & 0 \end{pmatrix} \underbrace{\mathbf{0}_{n \times \dots \times n \times (n-1)}}_k \end{pmatrix}, \quad 2 \leq k \leq \mu - 1,$$

and polynomials

$$g(X) = f(X) - f(x) - \sum_{1 \leq k \leq \mu-1} H_k(X - x)^k.$$

Theorem 8. *Let $\gamma_\mu = \gamma_\mu(g, x)$, if*

$$(3.23) \quad \|f(x)\| + \sum_{1 \leq k \leq \mu-1} \|H_k\| \left(\frac{d}{4\gamma_\mu^\mu} \right)^k < \frac{d^{\mu+1}}{2(4\gamma_\mu^\mu)^\mu \|\mathcal{A}^{-1}\|},$$

then f has μ zeros (counting multiplicities) in the ball of radius $\frac{d}{4\gamma_\mu^\mu}$ around x .

Proof. We have $g(x) = 0$, $Dg(x) = Df(x) - H_1 = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix}$. Moreover,

$$\Delta_k(g_n) = \Delta_k(f_n) - \Delta_k(f_n) = 0, \quad 2 \leq k \leq \mu - 1, \quad \Delta_\mu(g_n) = \Delta_\mu(f_n) \neq 0.$$

Therefore, $Dg(x)$ satisfies the normalized form, and x is a simple multiple root of g with multiplicity μ .

Let $R = \frac{d}{4\gamma_\mu^\mu(g, x)}$, we have

$$\begin{aligned} d_R(g, f) &= \max_{\|y-x\| \leq R} \|g(y) - f(y)\| \\ &= \max_{\|y-x\| \leq R} \|f(x) + \sum_{1 \leq k \leq \mu-1} H_k(X - x)^k\| \\ &\leq \|f(x)\| + \sum_{1 \leq k \leq \mu-1} \|H_k\| R^k \end{aligned}$$

$$= \|f(x)\| + \sum_{1 \leq k \leq \mu-1} \|H_k\| \left(\frac{d}{4\gamma_\mu^\mu} \right)^k.$$

If

$$\|f(x)\| + \sum_{1 \leq k \leq \mu-1} \|H_k\| \left(\frac{d}{4\gamma_\mu^\mu} \right)^k < \frac{d^{\mu+1}}{2(4\gamma_\mu^\mu)^\mu \|\mathcal{A}^{-1}\|},$$

then

$$d_R(g, f) < \frac{d^{\mu+1}}{2(4\gamma_\mu^\mu)^\mu \|\mathcal{A}^{-1}\|} = \frac{dR^\mu}{2\|\mathcal{A}^{-1}\|}.$$

By Theorem 7, we know that f has μ zeros (counting multiplicities) in the ball of radius $\frac{d}{4\gamma_\mu^\mu}$ around x . \square

3.3. Re-examining Double Simple Zeros. In what follows, we assume x is a simple double zero of f , $Df(x)$ satisfies the normalized form, and y is another zero of f . By expanding $f_n(y)$ at x and $\frac{\partial f_n(x)}{\partial X_1} = \dots = \frac{\partial f_n(x)}{\partial X_n} = \frac{\partial^2 f_n(x)}{\partial X_1^2} = 0$, we have:

$$f_n(y) = \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}} \zeta \eta + \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2} \eta^2 + \sum_{k \geq 3} \frac{D^k f_n(x) (y-x)^k}{k!}.$$

The nonzero terms are

$$C_{1,1} = \frac{\partial^2 f_n(x)}{\partial X_1 \partial \hat{X}}, \quad C_{0,2} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial \hat{X}^2}, \quad c_{1,1} = 2, \quad c_{0,2} = 1.$$

Hence, $p(d)$ has the following form:

$$(3.24) \quad p(d) = 1 - 2d^2 - 2d\sqrt{1-d^2} - d.$$

The smallest positive real root of $p(d)$ is

$$d \approx 0.2865.$$

Let y be another root of f , by Theorem 5, we have

$$\|y - x\| \geq \frac{d}{2\gamma_2^2}.$$

Example 1. Suppose we are given polynomials:

$$\begin{cases} f_1 = X_1^2 - \frac{1}{4}X_1 - \frac{1}{2}X_2, \\ f_2 = \frac{1}{2}X_1X_2. \end{cases}$$

It is easy to check that $x = (0, 0)$ is a simple double zero of $f = \{f_1, f_2\}$, and $(1/4, 0)$ is another zero of f , and we have

$$Df(x) = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Let $g(X) = f(W \cdot X)$, where $W = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$, then

$$(3.25) \quad \begin{cases} g_1 = \frac{4}{5}X_1^2 - \frac{4}{5}X_1X_2 + \frac{1}{5}X_2^2 + \frac{\sqrt{5}}{4}X_2, \\ g_2 = -\frac{1}{5}X_1^2 - \frac{3}{10}X_1X_2 + \frac{1}{5}X_2^2, \end{cases}$$

and $x = (0, 0)$ is a simple double zero of g , $y = \left(\frac{1}{2\sqrt{5}}, -\frac{1}{4\sqrt{5}}\right)$ is another zero. We have

$$Dg(x) = \begin{pmatrix} 0 & \frac{\sqrt{5}}{4} \\ 0 & 0 \end{pmatrix},$$

and $\ker Dg(x) = \text{span}\{v\}$, where $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$D^2g(x) = \left(\begin{pmatrix} \frac{8}{5} & -\frac{4}{3} \\ -\frac{2}{5} & -\frac{3}{10} \end{pmatrix} \quad \begin{pmatrix} -\frac{4}{3} & \frac{2}{5} \\ -\frac{3}{10} & \frac{3}{5} \end{pmatrix} \right).$$

Since

$$\frac{\partial g_1(x)}{\partial X_2} = \frac{\sqrt{5}}{4}, \quad \frac{1}{2} \frac{\partial^2 g_2}{\partial X_1^2} = \Delta_2 = -\frac{1}{5},$$

we have

$$\hat{\gamma}_2 = \max \left(1, \left\| \left(\frac{\partial g_1(x)}{\partial X_2} \right)^{-1} \cdot \frac{D^2g_1(x)}{2} \right\| \right) = \frac{4}{\sqrt{5}},$$

and

$$\gamma_{2,2} = \max \left(1, \left\| \frac{1}{\Delta_2(g_2)} \cdot \frac{D^2g_2(x)}{2} \right\| \right) = 1.$$

Therefore,

$$\gamma_2 = \frac{4}{\sqrt{5}} \approx 1.7888,$$

and the local separation bound we obtained according to Theorem 5 for $\mu = 2$ is

$$\|y - x\| \geq \frac{d}{2\gamma_2^2} \geq 0.0447.$$

Now let us estimate the local separation bound by the method in [8]. The invertible linear operator defined in [8] is

$$A(f, x, v) = Df(x) \cdot \frac{1}{2} D^2f(x)(v, \Pi_v),$$

where $v = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$, and

$$D^2f(x) = \left(\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \right).$$

Let $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{C}^2$, denote $A = A(f, x, v)$, we have

$$\begin{aligned} A\omega &= Df(x)\omega + \frac{1}{2} D^2f(x)(v, \Pi_v\omega) = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ &+ \frac{1}{2} \left(\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{4}{5}\omega_1 - \frac{2}{5}\omega_2 \\ -\frac{2}{5}\omega_1 + \frac{1}{5}\omega_2 \end{pmatrix} \\ &= \begin{pmatrix} \left(-\frac{1}{4} + \frac{8}{5\sqrt{5}}\right)\omega_1 + \left(-\frac{1}{2} - \frac{4}{5\sqrt{5}}\right)\omega_2 \\ -\frac{2}{5\sqrt{5}}\omega_1 + \frac{1}{5\sqrt{5}}\omega_2 \end{pmatrix}. \end{aligned}$$

We have

$$A^{-1} \frac{D^2 f(x)}{2} = \left(\begin{pmatrix} -\frac{4}{5} & -\frac{2(25+8\sqrt{5})}{10\sqrt{5}} \\ -\frac{8}{5} & -\frac{-25+32\sqrt{5}}{20\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{2(25+8\sqrt{5})}{10\sqrt{5}} & 0 \\ -\frac{-25+32\sqrt{5}}{20\sqrt{5}} & 0 \end{pmatrix} \right).$$

The computation of the norm of the tensor $A^{-1} \frac{D^2 f(x)}{2}$ is quite challenge. However, using our SOS certificates for global optima of polynomials and rational functions [19], we can verify that

$$\gamma_2(f, x) = \max \left(1, \left\| A^{-1} \frac{D^2 f(x)}{2} \right\| \right) \geq 3.1121.$$

Therefore, the local separation bound computed by the method in [8] satisfies

$$\frac{d}{2\gamma_2(f, x)^2} \leq 0.01546,$$

for $d \approx 0.2976$.

Remark 3. Although our d is smaller than the one obtained in [8], the value of γ_2 computed by our method could be smaller too. Therefore, as shown by Example 1, we might get better local separation bound.

Example 2. Suppose we are given polynomials:

$$\begin{cases} f_1 = \frac{64}{73}X_1^2 - \frac{48}{73}X_1X_2 + \frac{9}{73}X_2^2 + \frac{\sqrt{73}}{12}X_2, \\ f_2 = (8X_1 - 3X_2)^2(3X_1 + 8X_2). \end{cases}$$

Let $x = (0, 0)$ be a simple triple zero of $f = \{f_1, f_2\}$, and $y = (\frac{2}{\sqrt{73}}, -\frac{3}{4\sqrt{73}})$ be another zero of f , $\|y - x\| = 0.25$.

We have

$$Df(x) = \begin{pmatrix} 0 & \frac{\sqrt{73}}{12} \\ 0 & 0 \end{pmatrix},$$

$$\frac{\partial f_1(x)}{\partial X_2} = \frac{\sqrt{73}}{12},$$

$$\Delta_3(f_2) = \frac{1}{6} \frac{\partial^3 f_2(x)}{\partial X_1^3} - \frac{\partial^2 f_2(x)}{\partial X_1 \partial X_2} \cdot \left(\frac{\partial f_1(x)}{\partial X_2} \right)^{-1} \frac{1}{2} \frac{\partial^2 f_1(x)}{\partial X_1^2} = 192,$$

$$\hat{\gamma}_3(f, x) = \max \left(1, \left\| \left(\frac{\partial f_1}{\partial X_2} \right)^{-1} \cdot \frac{D^2 f_1(x)}{2} \right\| \right) = \frac{12}{\sqrt{73}},$$

$$\gamma_{3,2}(f, x) = \max_{2 \leq k \leq 3} \left(1, \left\| \frac{1}{\Delta_3(f_2)} \cdot \frac{D^k f_2(x)}{k!} \right\|^{\frac{1}{k-1}} \right) = \frac{\sqrt{73} \cdot 146^{\frac{1}{4}}}{24},$$

$$\gamma_3(f, x) = \max(\hat{\gamma}_3(f, x), \gamma_{3,2}(f, x)) = \frac{12}{\sqrt{73}} \approx 1.4045.$$

By Theorem 1, the local separation bound we obtain is

$$\|y - x\| \geq \frac{d}{2\gamma_3^3} \approx 0.01545.$$

For an approximate solution $x = (-4.1291 \cdot 10^{-8}, -2.9505 \cdot 10^{-8})$ obtained after applying twicely the modified Newton iterations defined by Algorithm 4.2 to an approximate zero $x = (-0.01, 0.01)$, we have

$$\gamma_3(g, x) = \max(\hat{\gamma}_3(g, x), \gamma_{3,2}(g, x)) = \frac{12}{\sqrt{73}} \approx 1.4045,$$

and

$$\|f(x)\| + \|H_1\| \frac{d}{4\gamma_3^3} + \|H_2\| \frac{d^2}{16\gamma_3^6} \approx 1.937364 \cdot 10^{-8} < \frac{d^4}{128\gamma_3^9 \|\mathcal{A}^{-1}\|} \approx 1.937370 \cdot 10^{-8}.$$

By Theorem 4, we can guarantee that f has three zeros (counting multiplicities) in the ball of radius $\frac{d}{4\gamma_3^3} \approx 0.0076$ around x .

We notice that

$$\left\| \frac{1}{\Delta_3(f_2)} \cdot \frac{D^2 f_2(x)}{2} \right\| = 0.00004935 \neq \left\| \frac{1}{\Delta_3(g_2)} \cdot \frac{D^2 g_2(x)}{2} \right\| = 0.00001467.$$

4. MODIFIED NEWTON ITERATIONS

For simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form (2.9), we define modified Newton iterations and show the quantified quadratic convergence if the approximate zeros are near the exact singular zeros. For a simple multiple zero of arbitrary large multiplicity whose Jacobian matrix may not have a normalized form, we perform unitary transformations and modified Newton iterations based on our previous work in [24], and show its non-quantified quadratic convergence for simple multiple zeros and the quantified convergence for simple triple zeros.

4.1. γ -theorem for Simple Double Zeros. Given an approximate zero z of f with associated simple double zero ξ such that $D\hat{f}(\xi)$ is invertible and

$$\frac{\partial f_i(\xi)}{\partial X_1} = 0, \quad \frac{\partial f_n(\xi)}{\partial X_i} = 0, \quad 1 \leq i \leq n, \quad \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \neq 0,$$

we aim to approximate ξ by applying modified Newton's method to z and iterating k times such that $\|N_f^k(z) - \xi\| < \epsilon$ for a given accuracy ϵ .

Algorithm 1 Modified Newton Iteration for Simple Double Zero

Input:

f : a polynomial system;

$z = (z_1, \hat{z})$: an approximate simple double zero of f ;

Output:

$N_f(z) = (N_2(z_1), N_1(\hat{z}))$: a refined solution after one iteration;

1: $N_1(\hat{z}) \leftarrow \hat{z} - D\hat{f}(z)^{-1}\hat{f}(z)$;

2: $\hat{y} \leftarrow N_1(\hat{z})$;

3: $z \leftarrow (z_1, \hat{y})$;

4: $N_2(z_1) \leftarrow z_1 - \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial f_n(z)}{\partial X_1}$;

Definition 4. For an approximate zero z of f with associated simple double zero ξ , let $\gamma_2 = \gamma_2(f, \xi)$, $u = \gamma_2^2 \|z - \xi\|$, we define the following rational functions:

$$\begin{aligned} b_{2,1}(u) &= \frac{(1-2u)^2 u}{[2(1-2u)^2 - 1](1-u)}, \\ b_{2,2}(u) &= \frac{u}{[2(1-2u)^2 - 1](1-u)}, \\ b_{2,3}(u) &= \frac{u(32u^6 - 144u^5 + 272u^4 - 288u^3 + 174u^2 - 52u + 5)}{(24u^3 - 36u^2 + 18u - 1)(-1+u)^3(8u^2 - 8u + 1)}, \\ b_{2,4}(u) &= \frac{(-1+2u)^3(-2+u)u}{(24u^3 - 36u^2 + 18u - 1)(-1+u)^3(8u^2 - 8u + 1)}. \end{aligned}$$

Theorem 9. Let ξ be a simple double zero of f .

(1) If $u < u_2 \approx 0.0418$, where u_2 is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 = 1,$$

then the output of Algorithm 4.1 satisfies:

$$\|N_f(z) - \xi\| < \|z - \xi\|.$$

(2) If $u < u'_2 \approx 0.0318$, where u'_2 is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 = \frac{1}{4},$$

then after applying k times of the iteration defined in Algorithm 4.1, we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k - 1} \|z - \xi\|.$$

Lemma 4. For $u \leq u_2$ we have:

$$(4.1) \quad \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \leq \frac{(1-2u)^2}{2(1-2u)^2 - 1}.$$

Proof. The Talyor's expansions of $\hat{f}(z)$ and $D\hat{f}(z)$ at ξ are

$$\hat{f}(z) = D\hat{f}(\xi)(\hat{z} - \hat{\xi}) + \sum_{k \geq 2} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i},$$

and

$$D\hat{f}(z) = D\hat{f}(\xi) + \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{1}{i!(k-i-1)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i-1}.$$

Since $D\hat{f}(\xi)^{-1}$ exists, we have

$$\begin{aligned} & D\hat{f}(\xi)^{-1} D\hat{f}(z) \\ &= I_{n-1} + D\hat{f}(\xi)^{-1} \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{1}{i!(k-i-1)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i-1} \\ &= I_{n-1} + B. \end{aligned}$$

Hence, we have

$$\begin{aligned}
\|B\| &= \left\| D\hat{f}(\xi)^{-1}D\hat{f}(z) - I_{n-1} \right\| \leq \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{k!}{i!(k-i-1)!} \hat{\gamma}_2^{k-1} \|z - \xi\|^{k-1} \\
&= \sum_{k \geq 2} k \cdot 2^{k-1} (\hat{\gamma}_2 \|z - \xi\|)^{k-1} \\
&\leq \frac{1}{(1 - 2\hat{\gamma}_2 \|z - \xi\|)^2} - 1 \\
&\leq \frac{1}{(1 - 2u)^2} - 1.
\end{aligned}$$

When $u < u_2$, $\|B\| < 1$, we have

$$\left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| = \left\| (I_{n-1} + B)^{-1} \right\| \leq \sum_{k=0}^{\infty} \|B\|^k \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1}$$

□

Lemma 5. *For $u \leq u_2$, we have:*

$$\begin{aligned}
\left\| N_1(\hat{z}) - \hat{\xi} \right\| &\leq \frac{\hat{\gamma}_2 \|\hat{z} - \hat{\xi}\|^2}{[2(1 - 2u)^2 - 1](1 - u)} + \frac{(1 - 2u)^2 \hat{\gamma}_2 |z_1 - \xi_1|^2}{[2(1 - 2u)^2 - 1](1 - u)} \\
&\leq b_{2,1}(u) |z_1 - \xi_1| + b_{2,2}(u) \|\hat{z} - \hat{\xi}\|.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\left\| N_1(\hat{z}) - \hat{\xi} \right\| \\
&= \left\| \hat{z} - \hat{\xi} - D\hat{f}(z)^{-1}\hat{f}(z) \right\| \\
&= \left\| D\hat{f}(z)^{-1} \left[D\hat{f}(z)(\hat{z} - \hat{\xi}) - \hat{f}(z) \right] \right\| \\
&= \left\| D\hat{f}(z)^{-1} \left[D\hat{f}(\xi)(\hat{z} - \hat{\xi}) + \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{k-i}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i} \right. \right. \\
&\quad \left. \left. - D\hat{f}(\xi)(\hat{z} - \hat{\xi}) - \sum_{k \geq 2} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i} \right] \right\| \\
&\leq \left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| \cdot \left\| D\hat{f}(\xi)^{-1} \sum_{k \geq 2} \sum_{i=0}^{k-2} \frac{k-i-1}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i} \right. \\
&\quad \left. - D\hat{f}(\xi)^{-1} \sum_{k \geq 2} \frac{1}{k!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^k} (z_1 - \xi_1)^k \right\| \\
&\leq \left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} \sum_{i=0}^{k-2} \frac{(k-i-1)k!}{i!(k-i)!} \hat{\gamma}_2^{k-1} \|z - \xi\|^{k-2} \|\hat{z} - \hat{\xi}\|^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k \geq 2} \hat{\gamma}_2^{k-1} |z_1 - \xi_1|^k \Big) \\
& \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} (k \cdot 2^{k-1} - 2^k + 1) \hat{\gamma}_2^{k-1} \|z - \xi\|^{k-2} \|\hat{z} - \hat{\xi}\|^2 \right. \\
& \quad \left. + \sum_{k \geq 2} \hat{\gamma}_2^{k-1} \|z - \xi\|^{k-2} |z_1 - \xi_1|^2 \right) \\
& \leq \frac{(1-2u)^2}{2(1-2u)^2 - 1} \cdot \left(\frac{\hat{\gamma}_2}{(1-u)(1-2u)^2} \|\hat{z} - \hat{\xi}\|^2 + \frac{\hat{\gamma}_2}{1-u} |z_1 - \xi_1|^2 \right) \\
& \leq b_{2,1}(u) |z_1 - \xi_1| + b_{2,2}(u) \|\hat{z} - \hat{\xi}\|.
\end{aligned}$$

□

Remark 4. The Newton iteration defined by N_1 operator works for any simple multiple zero of multiplicity $\mu \geq 2$ whose Jacobian matrix has a normalized form. It is clear from the proofs of Lemma 4 and Lemma 5, if we set $u = \gamma_\mu^\mu \|z - \xi\|$, then the conclusions of both lemmas still hold.

Let $z = (z_1, \hat{y})$, where $\hat{y} = N_1(\hat{z})$, then we have the following Talyor's expansion of $f_n(z)$ at ξ :

$$f_n(z) = \sum_{k \geq 2} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i}.$$

Lemma 6. When $u < u_2$, we have

$$\left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \leq \frac{(2u-1)^3}{24u^3 - 36u^2 + 18u - 1}.$$

Proof. We have

$$\begin{aligned}
& \left(\frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(z)}{\partial X_1^2} \\
& = 1 + \left(\frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \sum_{k \geq 3} \sum_{i=2}^k \frac{1}{(i-2)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \\
& = 1 + B,
\end{aligned}$$

where

$$\begin{aligned}
|B| & = \left| \left(\frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(z)}{\partial X_1^2} - 1 \right| \\
& \leq \sum_{k \geq 3} \sum_{i=2}^k \frac{k!}{(i-2)!(k-i)!} \gamma_{2,n}^{k-1} \|z - \xi\|^{k-2} \\
& = \sum_{k \geq 3} k(k-1) \cdot 2^{k-2} (\gamma_{2,n} \|z - \xi\|)^{k-3} \gamma_{2,n}^2 \|z - \xi\|
\end{aligned}$$

$$\leq \frac{-16u^3 + 24u^2 - 12u}{(2u-1)^3}.$$

When $u < u_2$, $|B| < 1$, we have

$$\left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| = |(1+B)^{-1}| \leq \sum_{k=0}^{\infty} |B|^k \leq \frac{(2u-1)^3}{24u^3 - 36u^2 + 18u - 1}.$$

□

Lemma 7. *When $u < u_2$, we have*

$$|N_2(z_1) - \xi_1| \leq b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)\|\hat{z} - \hat{\xi}\|.$$

Proof. For $u < u_2$, we have

$$\begin{aligned} & |N_2(z_1) - \xi_1| \\ &= \left| z_1 - \xi_1 - \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial f_n(z)}{\partial X_1} \right| \\ &= \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \left[\frac{\partial^2 f_n(z)}{\partial X_1^2} (z_1 - \xi_1) - \frac{\partial f_n(z)}{\partial X_1} \right] \right| \\ &= \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \left[\sum_{k \geq 2} \sum_{i=2}^k \frac{1}{(i-2)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i} \right. \right. \\ &\quad \left. \left. - \sum_{k \geq 2} \sum_{i=1}^k \frac{1}{(i-1)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i} \right] \right| \\ &\leq \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \cdot \left| \left(\frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \sum_{k \geq 2} \frac{1}{(k-1)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1 \partial \hat{X}^{k-1}} (\hat{y} - \hat{\xi})^{k-1} \right. \\ &\quad \left. - \left(\frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \sum_{k \geq 3} \sum_{i=3}^k \frac{i-2}{(i-1)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i} \right| \\ &\leq \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \cdot \left(\sum_{k \geq 3} \sum_{i=3}^k \frac{(i-2)k!}{(i-1)!(k-i)!} \gamma_{2,n}^{k-1} \|z - \xi\|^{k-2} |z_1 - \xi_1| \right. \\ &\quad \left. + \sum_{k \geq 2} k \gamma_{2,n}^{k-1} \|z - \xi\|^{k-2} \|\hat{y} - \hat{\xi}\| \right) \\ &\leq \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \cdot \left(\sum_{k \geq 3} \sum_{i=3}^k \frac{(i-2)k!}{(i-1)!(k-i)!} (\gamma_{2,n}^2 \|z - \xi\|)^{k-2} |z_1 - \xi_1| \right. \\ &\quad \left. + \sum_{k \geq 2} k (\gamma_{2,n}^2 \|z - \xi\|)^{k-2} \gamma_{2,n} \|\hat{y} - \hat{\xi}\| \right) \\ &\leq \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\sum_{k \geq 3} (k(k-3)2^{k-2} + k)u^{k-2}|z_1 - \xi_1| + \sum_{k \geq 2} ku^{k-2}\gamma_{2,n}\|\hat{y} - \hat{\xi}\| \right) \\
& \leq \left| \left(\frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \cdot \left(\frac{u(4u-3)}{(u-1)^2(2u-1)^3}|z_1 - \xi_1| + \frac{(2-u)\gamma_{2,n}}{(u-1)^2}\|\hat{y} - \hat{\xi}\| \right).
\end{aligned}$$

Then by Lemma 5 and Lemma 6, we have:

$$\begin{aligned}
& |N_2(z_1) - \xi_1| \\
& \leq \frac{u(32u^6 - 144u^5 + 272u^4 - 288u^3 + 174u^2 - 52u + 5)}{(24u^3 - 36u^2 + 18u - 1)(-1+u)^3(8u^2 - 8u + 1)}|z_1 - \xi_1| \\
& \quad + \frac{(-1+2u)^3(-2+u)u}{(24u^3 - 36u^2 + 18u - 1)(-1+u)^3(8u^2 - 8u + 1)}\|\hat{z} - \hat{\xi}\|. \\
& = b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)\|\hat{z} - \hat{\xi}\|.
\end{aligned}$$

□

Proof. Now we can complete the proof of Theorem 9:

(1) For $0 < u < u_2 \approx 0.0418$, we have

$$2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 < 1, \quad 2b_{2,2}(u)^2 + 2b_{2,4}(u)^2 < 1.$$

Hence, we have

$$\begin{aligned}
\|N_f(z) - \xi\|^2 & \leq \|N_1(\hat{z}) - \hat{\xi}\|^2 + |N_2(z_1) - \xi_1|^2 \\
& \leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|)^2 + (b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)\|\hat{z} - \hat{\xi}\|)^2 \\
& \leq (2b_{2,1}(u)^2 + 2b_{2,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{2,4}(u)^2)\|\hat{z} - \hat{\xi}\|^2 \\
& < \|z - \xi\|^2.
\end{aligned}$$

(2) For $0 < u < u'_2 \approx 0.0318$, we have

$$2b_{2,2}(u)^2 + 2b_{2,4}(u)^2 < 2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 < \frac{1}{4}.$$

Hence, we have

$$\begin{aligned}
\|N_f(z) - \xi\|^2 & \leq \|N_1(\hat{z}) - \hat{\xi}\|^2 + |N_2(z_1) - \xi_1|^2 \\
& \leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|)^2 + (b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)\|\hat{z} - \hat{\xi}\|)^2 \\
& \leq (2b_{2,1}(u)^2 + 2b_{2,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{2,4}(u)^2)\|\hat{z} - \hat{\xi}\|^2 \\
& \leq (2b_{2,1}(u)^2 + 2b_{2,3}(u)^2)\|z - \xi\|^2 \\
& \leq \frac{1}{4}\|z - \xi\|^2.
\end{aligned}$$

The following inequality is true for $k = 1$:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

For $k \geq 2$, assume by induction that

$$\|N_f^{k-1}(z) - \xi\| < \left(\frac{1}{2}\right)^{2^{k-1}-1} \|z - \xi\|.$$

Let $u^{(k-1)} = \gamma_2^2 \|N_f^{k-1}(z) - \xi\|$. For $0 < u < u'_2$, $k \geq 2$, we have $u^{(k-1)} < u = \gamma_2^2 \|z - \xi\|$ and $\frac{\sqrt{2b_{2,1}(u)^2 + 2b_{2,3}(u)^2}\gamma_2^2}{u}$ is increasing. Therefore, we have

$$\begin{aligned}
& \|N_f^k(z) - \xi\| \\
&= \|N_f(N_f^{k-1}(z)) - \xi\| \\
&< \frac{\sqrt{2b_{2,1}(u^{(k-1)})^2 + 2b_{2,3}(u^{(k-1)})^2}\gamma_2^2}{u^{(k-1)}} \|N_f^{k-1}(z) - \xi\|^2 \\
&< \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{2,3}(u)^2}\gamma_2^2}{u} \|N_f^{k-1}(z) - \xi\|^2 \\
&< \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{2,3}(u)^2}\gamma_2^2}{u} \left(\frac{1}{2}\right)^{2^{k-2}} \|z - \xi\|^2 \\
&= \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.
\end{aligned}$$

□

4.2. γ -theorem for Simple Triple Zeros. Given an approximate zero z of f with associated simple triple zero ξ such that $D\hat{f}(\xi)$ is invertible and

$$\frac{\partial f_i(\xi)}{\partial X_1} = 0, \quad \frac{\partial f_n(\xi)}{\partial X_i} = 0, \quad 1 \leq i \leq n, \quad \frac{\partial^2 f_n(\xi)}{\partial X_1^2} = 0,$$

but

$$\Delta_3(f_n) = \frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^3} - \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(\xi)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(\xi)}{\partial X_1^2} \neq 0.$$

We aim to approximate ξ by applying modified Newton's method to z and iterating k times such that $\|N_f^k(z) - \xi\| < \epsilon$ for a given accuracy ϵ .

Let us define the differential operator L_3 ,

$$L_3(f_n)(z) = \frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_1^3} - \frac{\partial^2 f_n(z)}{\partial X_1 \partial \hat{X}} \cdot D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2},$$

then we have $L_3(f_n)(\xi) = \Delta_3(f_n)$. As $\Delta_3(f_n) \neq 0$, and z is near to ξ , we can assume that $L_3(f_n)(z) \neq 0$. Moreover, we define the differential operator Γ_1 such that

$$\Gamma_1(f_n)(z) = \frac{1}{6} \frac{\partial^2 f_n(z)}{\partial X_1^2} - \frac{\partial f_n(z)}{\partial \hat{X}} \cdot D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2}.$$

Algorithm 2 Modified Newton Iteration for Simple Triple Zero**Input:** f : a polynomial system; $z = (z_1, \hat{z})$: an approximate simple triple zero of f ;**Output:** $N_f(z) = (N_2(z_1), N_1(\hat{z}))$: a refined solution after one iteration;1: $N_1(\hat{z}) \leftarrow \hat{z} - D\hat{f}(z)^{-1}\hat{f}(z)$;2: $\hat{y} \leftarrow N_1(\hat{z})$;3: $z \leftarrow (z_1, \hat{y})$;4: $N_2(z_1) \leftarrow z_1 - (L_3(f_n)(z))^{-1} \cdot \Gamma_1(f_n)(z)$;

Definition 5. For an approximate root z of ξ , let $u = \gamma_3^3 \|z - \xi\|$. We define the following rational functions:

$$\begin{aligned}
 a_2(u) &= \frac{1}{[2(1-2u)^2 - 1](1-2u)}, \\
 a_3(u) &= \frac{(2u-1)^4(8u^2-8u+1)}{128u^6 - 384u^5 + 464u^4 - 320u^3 + 136u^2 - 30u + 1}, \\
 b_{3,3}(u) &= \frac{-a_3(u)}{3(2u-1)^4(8u^2-8u+1)^2(u-1)^4} \cdot (3072u^{12} - 25088u^{11} \\
 &\quad + 92480u^{10} - 202336u^9 + 289640u^8 - 282020u^7 + 188614u^6 \\
 &\quad - 85997u^5 + 26342u^4 - 5368u^3 + 702u^2 - 42u), \\
 b_{3,4}(u) &= \frac{a_3(u)(16u^6 - 72u^5 + 130u^4 - 106u^3 + 42u^2 - 9u)}{3(8u^2 - 8u + 1)^2(u-1)^4(2u-1)}.
 \end{aligned}$$

Theorem 10. Let ξ be a simple triple zero of f .

(1) If $u < u_3 \approx 0.0222$, where u_3 is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 = 1,$$

then the output of Algorithm 4.2 satisfies:

$$\|N_f(z) - \xi\| < \|z - \xi\|.$$

(2) If $u < u'_3 \approx 0.0154$, where u'_3 is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 = \frac{1}{4},$$

then after k times of iteration we have

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

According to Remark 4, similar to Lemma 4 and Lemma 5, for $u \leq u_3 \approx 0.0222$, we have:

$$\left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \leq \frac{(1-2u)^2}{2(1-2u)^2 - 1},$$

and

$$\left\| N_1(\hat{z}) - \hat{\xi} \right\| \leq b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|.$$

Let $z = (z_1, \hat{y})$, where $\hat{y} = N_1(\hat{z})$, we have the following Talyor's expansion of $f_n(z)$ at ξ :

$$f_n(z) = \sum_{k \geq 2} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i}.$$

Lemma 8. *When $u < u_3$, we have*

$$\left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| \leq a_2(u) \gamma_3.$$

Proof. When $u < u_3$, we have

$$\left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \leq \frac{(1-2u)^2}{2(1-2u)^2 - 1}.$$

Then, it is clear that

$$\begin{aligned} & \left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| \\ &= \left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \sum_{k \geq 2} \sum_{i=2}^k \frac{1}{(i-2)!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \right\| \\ &= \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) D\hat{f}(\xi)^{-1} \right. \\ & \quad \cdot \left. \frac{1}{2} \sum_{k \geq 2} \sum_{i=2}^k \frac{1}{(i-2)!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \right\| \\ &\leq \frac{1}{2} \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} \sum_{i=2}^k \frac{k!}{(i-2)!(k-i)!} \hat{\gamma}_3^{k-1} \|z - \xi\|^{k-2} \right) \\ &= \frac{1}{2} \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} k(k-1) (2\hat{\gamma}_3 \|z - \xi\|)^{k-2} \hat{\gamma}_3 \right) \\ &\leq \frac{(1-2u)^2}{2(1-2u)^2 - 1} \cdot \frac{1}{(1-2u)^3} \gamma_3 \\ &= a_2(u) \gamma_3. \end{aligned}$$

□

Lemma 9. *When $u < u_3$, we have*

$$\left\| L_3(f_n)(z)^{-1} \Delta_3(f_n) \right\| \leq a_3(u).$$

Proof. By the Taylor's expansion of f_n at ξ , we have:

$$\begin{aligned} \frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_1^3} &= \frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^3} + \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \frac{1}{(i-3)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-3} (\hat{y} - \hat{\xi})^{k-i}, \\ \frac{\partial^2 f_n(z)}{\partial X_1 \partial \hat{X}} &= \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial \hat{X}} + \frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (z_1 - \xi_1) + \frac{\partial^3 f_n(\xi)}{\partial X_1 \partial \hat{X}^2} (\hat{y} - \hat{\xi}) \end{aligned}$$

$$+ \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{1}{(i-1)!(k-i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i-1}.$$

Then we have:

$$\begin{aligned} & \Delta_3(f_n)^{-1} L_3(f_n)(z) \\ &= 1 + \Delta_3(f_n)^{-1} [L_3(f_n)(z) - \Delta_3(f_n)] \\ &= 1 + B, \end{aligned}$$

where

$$\begin{aligned} \|B\| = & \left\| L_3(f_n)(\xi)^{-1} \cdot D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \cdot \left[\frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (z_1 - \xi_1) + \frac{\partial^3 f_n(\xi)}{\partial X_1 \partial \hat{X}^2} (\hat{y} - \hat{\xi}) \right. \right. \\ & + \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{1}{(i-1)!(k-i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i-1} \left. \right] \\ & - \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \frac{1}{(i-3)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-3} (\hat{y} - \hat{\xi})^{k-i} \left. \right\|. \end{aligned}$$

By Lemma 8, we have

$$\begin{aligned} \|B\| \leq & 12a_2(u) \cdot \gamma_3^3 \cdot \|z - \xi\| + a_2(u) \sum_{k \geq 4} \sum_{i=1}^{k-1} \binom{k-2}{i-1} k(k-1) \cdot \gamma_3^k \|z - \xi\|^{k-2} \\ & + \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^{k-1} \binom{k-3}{i-3} (k-1)(k-2)(k-3) \cdot \gamma_3^{k-1} \|z - \xi\|^{k-3} \\ \leq & 12a_2(u) \cdot \gamma_3^3 \cdot \|z - \xi\| + a_2(u) \sum_{k \geq 4} 2^{k-2} \cdot k \cdot (k-1) \cdot (\gamma_3^2)^{k-2} \|z - \xi\|^{k-2} \\ & + \frac{1}{6} \sum_{k \geq 4} 2^{k-3} \cdot (k-1) \cdot (k-2) \cdot (k-3) \cdot (\gamma_3^3)^{k-3} \|z - \xi\|^{k-3} \\ \leq & a_2(u) \left(12u + \sum_{k \geq 4} k(k-1) 2^{k-2} u^{k-2} \right) + \frac{1}{6} \sum_{k \geq 4} (k-1)(k-2)(k-3) 2^{k-3} u^{k-3} \\ = & \frac{2u(16u^2 - 20u + 7)}{(2u-1)^4(8u^2 - 8u + 1)}. \end{aligned}$$

When $u < u_3$, $\|B\| < 1$, we have

$$\|L_3(f_n)(z)^{-1} \Delta_3(f_n)\| \leq \|(1+B)^{-1}\| \leq a_3(u).$$

□

Lemma 10. *When $u < u_3$, we have*

$$|N_2(z_1) - \xi_1| \leq b_{3,3}(u)(u)|z_1 - \xi_1| + b_{3,4}(u)(u)\|\hat{z} - \hat{\xi}\|.$$

Proof. We have

$$\begin{aligned} |N_2(z_1) - \xi_1| &= \left| z_1 - \xi_1 - [L_3(f_n)(z)]^{-1} \Gamma_1(f_n)(z) \right| \\ &= \left| L_3(f_n)(z)^{-1} [L_3(f_n)(z)(z_1 - \xi_1) - \Gamma_1(f_n)(z)] \right| \end{aligned}$$

$$= |L_3(f_n)(z)^{-1} \Delta_3(f_n)| \cdot |\Delta_3(f_n)^{-1} [L_3(f_n)(z)(z_1 - \xi_1) - \Gamma_1(f_n)(z)]|.$$

From the Taylor expansions of $\frac{\partial^3 f_n(z)}{\partial X_1^3}, \frac{\partial^2 f_n(z)}{\partial X_1^2}, \frac{\partial^2 f_n(z)}{\partial X_1 \partial \hat{X}}, \frac{\partial f_n(z)}{\partial \hat{X}}$ at ξ , we have

$$\begin{aligned} & L_3(f_n)(z)(z_1 - \xi_1) - \Gamma_1(f_n)(z) \\ &= -\frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (\hat{y} - \hat{\xi}) - \frac{1}{6} \sum_{k \geq 4} \frac{1}{(k-2)!} \frac{\partial^k f_n(\xi)}{\partial X_1^2 \partial \hat{X}^{k-2}} (\hat{y} - \hat{\xi})^{k-2} \\ & \quad + \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \frac{i-3}{(i-2)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \\ & \quad - D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \left[\frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (z_1 - \xi_1)^2 - \frac{\partial^2 f_n(\xi)}{\partial \hat{X}^2} (\hat{y} - \hat{\xi}) - \frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial \hat{X}^3} (\hat{y} - \hat{\xi})^2 \right. \\ & \quad + \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{i-1}{i!(k-i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i-1} \\ & \quad \left. - \sum_{k \geq 4} \frac{1}{(k-1)!} \frac{\partial^k f_n(\xi)}{\partial \hat{X}^k} (\hat{y} - \hat{\xi})^{k-1} \right]. \end{aligned}$$

Then we have

$$\begin{aligned} & \left\| \frac{1}{|\Delta_3(f_n)|} \cdot \left(-\frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (\hat{y} - \hat{\xi}) \right) \right\| + \left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| \cdot \left\| \frac{1}{|\Delta_3(f_n)|} \cdot \frac{\partial^2 f_n(\xi)}{\partial \hat{X}^2} (\hat{y} - \hat{\xi}) \right\| \\ & \leq \gamma_3^2 \cdot \|(\hat{y} - \hat{\xi})\| + 2a_2(u) \gamma_3^2 \cdot \|(\hat{y} - \hat{\xi})\| \\ & \leq (1 + 2a_2(u)) \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} \gamma_3^{k+1} \|z - \xi\|^{k-1} |z_1 - \xi_1| \right. \\ & \quad \left. + \sum_{k \geq 2} (k \cdot 2^{k-1} - 2^k + 1) \gamma_3^{k+1} \|z - \xi\|^{k-1} \|\hat{z} - \hat{\xi}\| \right) \\ & \leq (1 + 2a_2(u)) \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\sum_{k \geq 2} (\gamma_3^3 \|z - \xi\|)^{k-1} |z_1 - \xi_1| \right. \\ & \quad \left. + \sum_{k \geq 2} (k \cdot 2^{k-1} - 2^k + 1) (\gamma_3^3 \|z - \xi\|)^{k-1} \|\hat{z} - \hat{\xi}\| \right) \\ & \leq \frac{u(2u-1)^2(1+2a_2(u))}{(8u^2-8u+1)(1-u)} |z_1 - \xi_1| + \frac{u(1+2a_2(u))}{(8u^2-8u+1)(1-u)} \|\hat{z} - \hat{\xi}\|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \frac{1}{|\Delta_3(f_n)|} \cdot \left\| \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \frac{i-3}{(i-2)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \right\| \\ & \leq \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \frac{k! \cdot (i-3)}{(i-2)!(k-i)!} \gamma_3^{k-1} \|z - \xi\|^{k-3} \cdot |z_1 - \xi_1| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^k \binom{k-3}{i-3} \cdot k \cdot (k-1) \cdot (k-2) \cdot \gamma_3^{3k-9} \cdot \|z - \xi\|^{k-3} \cdot |z_1 - \xi_1| \\
&\leq \frac{1}{6} \sum_{k \geq 4} k \cdot (k-1) \cdot (k-2) \cdot 2^{k-3} \cdot (\gamma_3^3 \|z - \xi\|)^{k-3} |z_1 - \xi_1| \\
&\leq \frac{1}{6} \sum_{k \geq 4} k \cdot (k-1) \cdot (k-2) \cdot 2^{k-3} \cdot u^{k-3} |z_1 - \xi_1| \\
&= \frac{-8u(2u^3 - 4u^2 + 3u - 1)}{(2u-1)^4} |z_1 - \xi_1|.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{|\Delta_3(f_n)|} \cdot \left\| \frac{1}{6} \sum_{k \geq 4} \frac{1}{(k-2)!} \frac{\partial^k f_n(\xi)}{\partial X_1^2 \partial \hat{X}^{k-2}} (\hat{y} - \hat{\xi})^{k-2} \right\| \\
&\leq \frac{1}{6} \sum_{k \geq 4} k(k-1) \gamma_3^{k-1} \|z - \xi\|^{k-3} \|\hat{y} - \hat{\xi}\| \\
&\leq \frac{1}{6} \sum_{k \geq 4} k(k-1) (\gamma_3^3 \|z - \xi\|)^{k-3} \|\hat{y} - \hat{\xi}\| \\
&\leq \frac{-u(3u^2 - 8u + 6)}{3(u-1)^3} \|\hat{y} - \hat{\xi}\| \\
&\leq \frac{(2u-1)^2 u^2 (3u^2 - 8u + 6)}{3(8u^2 - 8u + 1)(u-1)^4} |z_1 - \xi_1| + \frac{u^2(3u^2 - 8u + 6)}{3(8u^2 - 8u + 1)(u-1)^4} \|\hat{z} - \hat{\xi}\|
\end{aligned}$$

By Lemma 8, we have

$$\begin{aligned}
&\frac{\left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\|}{|\Delta_3(f_n)|} \left\| \frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial X_1^2 \partial \hat{X}} (z_1 - \xi_1)^2 \right\| \leq 3a_2(u) \gamma_3^3 \|z - \xi\| |z_1 - \xi_1| \\
&\leq 3a_2(u) u |z_1 - \xi_1|.
\end{aligned}$$

By Lemma 8, we have

$$\begin{aligned}
&\frac{\left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\|}{|\Delta_3(f_n)|} \left\| \frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial \hat{X}^3} (\hat{y} - \hat{\xi})^2 \right\| \\
&\leq 3a_2(u) \gamma_3^3 \|z - \xi\| \|\hat{y} - \hat{\xi}\| \\
&\leq \frac{3a_2(u) u^2}{[2(1-2u)^2 - 1](1-u)} \|\hat{z} - \hat{\xi}\| + \frac{3a_2(u) u^2 (1-2u)^2}{[2(1-2u)^2 - 1](1-u)} |z_1 - \xi_1|.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{\left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\|}{|\Delta_3(f_n)|} \cdot \left\| \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{i-1}{i!(k-i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial \hat{X}^{k-i}} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i-1} \right\| \\
&\leq a_2(u) \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{k! \cdot (i-1)}{i!(k-i-1)!} \gamma_3^k \|z - \xi\|^{k-2} \cdot |z_1 - \xi_1| \\
&\leq a_2(u) \sum_{k \geq 4} (k-1) \cdot (k-3) \cdot 2^k \cdot u^{k-2} |z_1 - \xi_1|
\end{aligned}$$

$$\leq \frac{16u^2(2u-3)a_2(u)}{(u-1)^3} |z_1 - \xi_1|.$$

By Lemma 8, we have

$$\begin{aligned} & \frac{\left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial \hat{X}_1^2} \right\|}{|\Delta_3(f_n)|} \cdot \left\| \sum_{k \geq 4} \frac{1}{(k-1)!} \frac{\partial^k f_n(\xi)}{\partial \hat{X}^k} (\hat{y} - \hat{\xi})^{k-1} \right\| \\ & \leq a_2(u) \sum_{k \geq 4} k \gamma_3^k \|z - \xi\|^{k-2} \|\hat{y} - \hat{\xi}\| \\ & \leq \frac{-u^2(3u-4)a_2(u)}{(u-1)^2} \|\hat{y} - \hat{\xi}\| \\ & \leq \frac{(2u-1)^2 u^3 (3u-4)a_2(u)}{(8u^2-8u+1)(u-1)^3} |z_1 - \xi_1| + \frac{u^3(3u-4)a_2(u)}{(8u^2-8u+1)(u-1)^3} \|\hat{z} - \hat{\xi}\|. \end{aligned}$$

Finally, by Lemma 9 and the above estimations, we have

$$\begin{aligned} & |N_2(z_1) - \xi_1| \\ & \leq a_3(u) \cdot \left(\left(\frac{u(2u-1)^2(1+2a_2(u))}{(8u^2-8u+1)(1-u)} + \frac{16u^2(2u-3)a_2(u)}{(u-1)^3} \right. \right. \\ & \quad + \frac{-8u(2u^3-4u^2+3u-1)}{(2u-1)^4} + \frac{3u^2(1-2u)^2 a_2(u)}{[2(1-2u)^2-1](1-u)} + 3ua_2(u) + \\ & \quad \left. \frac{(2u-1)^2 u^2 (3u^2-8u+6)}{3(8u^2-8u+1)(u-1)^4} + \frac{(2u-1)^2 u^3 (3u-4)a_2(u)}{(8u^2-8u+1)(u-1)^3} \right) \cdot |z_1 - \xi_1| \\ & \quad + \left(\frac{u(1+2a_2(u))}{(8u^2-8u+1)(1-u)} + \frac{3u^2 a_2(u)}{[2(1-2u)^2-1](1-u)} + \right. \\ & \quad \left. \frac{u^2(3u^2-8u+6)}{3(8u^2-8u+1)(u-1)^4} + \frac{u^3(3u-4)a_2(u)}{(8u^2-8u+1)(u-1)^3} \right) \cdot \|\hat{z} - \hat{\xi}\| \\ & = b_{3,3}(u) |z_1 - \xi_1| + b_{3,4}(u) \|\hat{z} - \hat{\xi}\|. \end{aligned}$$

□

Proof. Now we can complete the proof of Theorem 10.

(1) For $u < u_3 \approx 0.0222$, it is true that

$$2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 < 1, \quad 2b_{2,2}(u)^2 + 2b_{3,4}(u)^2 < 1.$$

Therefore, we have

$$\begin{aligned} & \|N_f(z) - \xi\|^2 \\ & \leq \|N_1(\hat{z}) - \hat{\xi}\|^2 + |N_2(z_1) - \xi_1|^2 \\ & \leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|)^2 + (b_{3,3}(u)|z_1 - \xi_1| + b_{3,4}(u)\|\hat{z} - \hat{\xi}\|)^2 \\ & \leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{3,4}(u)^2)\|\hat{z} - \hat{\xi}\|^2 \\ & < \|z - \xi\|^2. \end{aligned}$$

(2) For $u < u'_3 \approx 0.0154$, it is true that

$$2b_{2,2}(u)^2 + 2b_{3,4}(u)^2 < 2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 < \frac{1}{4}.$$

Hence, we have

$$\begin{aligned}
\|N_f(z) - \xi\|^2 &\leq \|N_1(\hat{z}) - \hat{\xi}\|^2 + |N_2(z_1) - \xi_1|^2 \\
&\leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|)^2 + (b_{3,3}(u)|z_1 - \xi_1| + b_{3,4}(u)\|\hat{z} - \hat{\xi}\|)^2 \\
&\leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{3,4}(u)^2)\|\hat{z} - \hat{\xi}\|^2 \\
&\leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2) \|z - \xi\|^2 \\
&\leq \frac{1}{4} \|z - \xi\|^2.
\end{aligned}$$

Hence, the following inequality is true for $k = 1$:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

For $k \geq 2$, assume by induction that

$$\|N_f^{k-1}(z) - \xi\| < \left(\frac{1}{2}\right)^{2^{k-1}-1} \|z - \xi\|.$$

Let $u^{(k-1)} = \gamma_3^3 \|N_f^{k-1}(z) - \xi\|$. For $0 < u < u'_3$, we have $u^{(k-1)} < u = \gamma_3^3 \|z - \xi\|$ and $\frac{\sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2}\gamma_3^3}{u}$ is increasing. Therefore, we have

$$\begin{aligned}
&\|N_f^k(z) - \xi\| \\
&= \|N_f(N_f^{k-1}(z)) - \xi\| \\
&< \frac{\sqrt{2b_{2,1}(u^{(k-1)})^2 + 2b_{3,3}(u^{(k-1)})^2}\gamma_3^3}{u^{(k-1)}} \|N_f^{k-1}(z) - \xi\|^2 \\
&< \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2}\gamma_3^3}{u} \|N_f^{k-1}(z) - \xi\|^2 \\
&< \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2}\gamma_3^3}{u} \left(\frac{1}{2}\right)^{2^k-2} \|z - \xi\|^2 \\
&= \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.
\end{aligned}$$

□

4.3. Simple Multiple Zeros. For simple double zeros and simple triple zeros of f , we have defined modified Newton iterations based on the first, second and third order differential operators computed at the approximate solutions, and provided quantified criterions to guarantee its quadratic convergence. Although it is possible to extend the modified Newton iterations defined in Algorithm 4.1, 4.2 to simple multiple zeros of higher multiplicities, the iterations are only defined for systems whose Jacobian matrix at the exact multiple zero has a normalized form (2.9), they might be of limited applications.

In order to refining an approximate simple singular zero whose Jacobian matrix has corank one but it does not have a normalized form (2.9), in Algorithm 4.3, we perform the unitary transformations to both variables and equations defined at the approximate simple singular solutions, then we define the modified Newton

iterations based on our previous work in [24]. We show firstly its non-quantified quadratic convergence for simple multiple zeros of higher multiplicities, and then its quantified convergence for simple triple zero.

Algorithm 3 Modified Newton Iteration for Simple Multiple Zeros

Input:

f : a polynomial system;
 z : an approximate simple multiple zero;
 μ : the multiplicity;

Output:

$N_f(z)$: a refined solution after one iteration;

- 1: $Df(z) = U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot V^*$, $W_{\dagger} = (v_n, v_1, \dots, v_{n-1})$;
 - 2: $f(X) \leftarrow U^* \cdot f(W_{\dagger} \cdot X)$, $z \leftarrow W_{\dagger}^* z$;
 - 3: $N_1(\hat{f}, \hat{z}) \leftarrow \hat{z} - D\hat{f}(z)^{-1}\hat{f}(z)$, $y = (y_1, \hat{y}) \leftarrow (z_1, N_1(\hat{f}, \hat{z}))$;
 - 4: $Df(z) = U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot V^*$, $W_{\dagger} = (v_n, v_1, \dots, v_{n-1})$;
 - 5: $g(X) \leftarrow U^* \cdot f(W_{\dagger} \cdot X)$, $w = (w_1, \hat{w}) \leftarrow W_{\dagger}^* y$;
 - 6: $N_2(g_n, w) \leftarrow w_1 - \frac{1}{\mu} \Delta_{\mu}(g_n)^{-1} \Delta_{\mu-1}(g_n)$, $x = (x_1, \hat{x}) \leftarrow (N_2(g_n, w), \hat{w})$;
 - 7: $N_f(z) \leftarrow W_{\dagger} \cdot W_{\dagger}^* \cdot x$.
-

Theorem 11. *Given an approximate zero z of a system f associated to a simple multiple zero ξ of multiplicity μ and satisfying $f(\xi) = 0$, $\dim \ker Df(\xi) = 1$. Suppose*

$$\hat{\gamma}_{\mu}(f, z) \|z - \xi\| < \frac{1}{2},$$

where

$$\hat{\gamma}_{\mu}(f, z) = \max \left\{ 1, \sup_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\|^{\frac{1}{k-1}} \right\},$$

then the refined singular solution $N_f(z)$ returned by Algorithm 4.3 satisfies

$$(4.2) \quad \|N_f(z) - \xi\| = O(\|z - \xi\|^2).$$

In what follows, we give quantitative analysis of the convergency of the first five steps in Algorithm 4.3. For Step 6, we show its non-quantified quadratic convergence first, and then show its quantified convergency for simple triple zeros, which can be generalized naturally to simple multiple zeros of higher multiplicities.

In the second step of Algorithm 4.3, we perform the unitary transformations to both variables and equations according to the singular value decomposition of the Jacobian matrix $Df(z)$. Since $\hat{\gamma}_{\mu}(f, z)$ and the Euclidean distance between zeros ξ and z do not change under the unitary transformation, in what follows, for simplicity, after the first two steps, we use the same notations for f, ξ, z , i.e.,

$$(4.3) \quad \xi \leftarrow W_{\dagger}^* \xi, \quad z \leftarrow W_{\dagger}^* z, \quad f(X) \leftarrow U^* \cdot f(W_{\dagger} \cdot X).$$

Claim 5. *After the first two steps in Algorithm 4.3, we have*

$$(4.4) \quad Df(z) = \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix},$$

where Σ_{n-1} is a nonsingular diagonal matrix. Moreover, we have $\sigma_n \leq L\|z - \xi\|$, where L is the Lipschitz constant of the function $Df(X)$.

Proof. According to the chain rule, we have

$$\begin{aligned} Df(z) &= U^* \cdot U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot V^* \cdot W_{\dagger} \\ &= \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix}. \end{aligned}$$

Furthermore, since $\dim \ker Df(\xi) = 1$, the following perturbation theorem about the singular values can be found in [12, 13],

$$\sigma_n \leq \|Df(z) - Df(\xi)\| \leq L\|z - \xi\|.$$

□

Claim 6. After running the first three steps in Algorithm 4.3, suppose $\hat{\gamma}_\mu(f, z)\|z - \xi\| < \frac{1}{2}$, we have

$$(4.5) \quad \|\hat{y} - \hat{\xi}\| \leq \frac{1}{1 - \hat{\gamma}_\mu(f, z)\|z - \xi\|} \hat{\gamma}_\mu(f, z)\|z - \xi\|^2,$$

and

$$(4.6) \quad \|\hat{f}(y)\| \leq \frac{4\|D\hat{f}(z)\|}{1 - 2\hat{\gamma}_\mu(f, z)\|z - \xi\|} \hat{\gamma}_\mu(f, z)\|z - \xi\|^2,$$

where

$$(4.7) \quad y = (y_1, \hat{y}) \leftarrow (z_1, N_1(\hat{f}, \hat{z})).$$

Proof. According to Claim 5, we have $\frac{\partial \hat{f}(z)}{\partial X_1} = 0$ and $D\hat{f}(z) = \Sigma_{n-1}$ is an invertible diagonal matrix. Therefore, by the Taylor expansion of \hat{f} at z , we have

$$\begin{aligned} 0 &= \hat{f}(\xi) = \hat{f}(z) + D\hat{f}(z)(\hat{\xi} - \hat{z}) + \sum_{k \geq 2} \frac{D^k \hat{f}(z)}{k!} (\xi - z)^k, \\ 0 &= D\hat{f}(z)^{-1} \hat{f}(z) + \hat{\xi} - \hat{z} + \sum_{k \geq 2} D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} (\xi - z)^k. \end{aligned}$$

Then we have

$$\begin{aligned} \|N_1(\hat{f}, z) - \hat{\xi}\| &\leq \sum_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \|\xi - z\|^k \\ &\leq \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-1} \|\xi - z\|^k \\ &\leq \hat{\gamma}_\mu(f, z)\|\xi - z\|^2 \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-2} \|\xi - z\|^{k-2} \\ &\leq \frac{1}{1 - \hat{\gamma}_\mu(f, z)\|\xi - z\|} \hat{\gamma}_\mu(f, z)\|\xi - z\|^2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|\hat{f}(y)\| &= \left\| \hat{f}(z) + D\hat{f}(z)(\hat{y} - \hat{z}) + \sum_{k \geq 2} \frac{D^k \hat{f}(z)}{k!} (y - z)^k \right\| \\
&\leq \|D\hat{f}(z)\| \sum_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \|y - z\|^k \\
&\leq \|D\hat{f}(z)\| \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-1} \|y - z\|^k \\
&\leq \|D\hat{f}(z)\| \hat{\gamma}_\mu(f, z) \|y - z\|^2 \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-2} \|y - z\|^{k-2} \\
&\leq \frac{4\|D\hat{f}(z)\|}{1 - 2\hat{\gamma}_\mu(f, z)\|\xi - z\|} \hat{\gamma}_\mu(f, z) \|\xi - z\|^2,
\end{aligned}$$

where

$$\|y - z\| = \|\hat{y} - \hat{z}\| \leq \|\hat{y} - \hat{\xi}\| + \|\hat{z} - \hat{\xi}\| \leq \frac{\hat{\gamma}_\mu(f, z)\|\xi - z\|}{1 - \hat{\gamma}_\mu(f, z)\|\xi - z\|} \|\xi - z\| + \|\xi - z\| \leq 2\|\xi - z\|,$$

and

$$\hat{\gamma}_\mu(f, z)\|y - z\| \leq 2\hat{\gamma}_\mu(f, z)\|\xi - z\| < 1.$$

□

Let $(\text{span}_{\mathbb{C}}\{v_n(z)\}, \text{span}_{\mathbb{C}}\{u_n(z)\})$ be a pair of singular subspaces of $Df(z)$ corresponding to its smallest singular value σ_n , and $\delta = \sigma_{n-1} - \sigma_n = O(1)$. If

$$(4.8) \quad \|Df(y) - Df(z)\|_F \leq \frac{\delta}{5},$$

which could be satisfied in general since y is close to z and $\delta = O(1)$, then according to [13, Theorem 8.6.5] or [43, Theorem 6.4], we have

$$(4.9) \quad \|v_n(y) - v_n(z)\|_F \leq 4 \frac{\|Df(y) - Df(z)\|_F}{\delta} \leq 4 \frac{L\|y - z\|}{\delta} \leq 8 \frac{L\|\xi - z\|}{\delta},$$

and

$$(4.10) \quad \|u_n(y) - u_n(z)\|_F \leq 4 \frac{\|Df(y) - Df(z)\|_F}{\delta} \leq 4 \frac{L\|y - z\|}{\delta} \leq 8 \frac{L\|\xi - z\|}{\delta},$$

where $(\text{span}_{\mathbb{C}}\{v_n(y)\}, \text{span}_{\mathbb{C}}\{u_n(y)\})$ is a pair of singular subspaces of $Df(y)$ corresponding to its smallest singular value.

In what follows, for the sake of simplicity, we always assume (4.8) is satisfied and set

$$L \leftarrow \frac{8L}{\delta}.$$

According to (4.4), we know that $v_n(z) = (1, 0, \dots, 0)^T$ and $u_n(z) = (0, \dots, 0, 1)$ generate a pair of singular subspaces of $Df(z)$ corresponding to its smallest singular value σ_n .

Let

$$W_{\ddagger} = (v_n(y), v_1(y), \dots, v_{n-1}(y)) = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},$$

and $v_n(y) = (W_1, W_3)^T$, by (4.9), we have

$$(4.11) \quad |W_1| \geq 1 - L\|\xi - z\|, \quad \|W_3\| \leq L\|\xi - z\|.$$

Since W_{\dagger} is a unitary matrix, we have

$$(4.12) \quad \|W_2\| \leq L\|\xi - z\|, \quad \|W_4\| \leq \|W_{\dagger}\| = 1.$$

Let $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ and $u_n(y) = (U_3, U_4)$, by (4.9), we have

$$(4.13) \quad \|U_3^*\| \leq L\|\xi - z\|.$$

It is clear that Step 4 and Step 5 in Algorithm 4.3 are used to normalize the Jacobian matrix at the approximate solution y again after running the Newton iteration for \hat{z} , i.e., we have

$$Df(w) = \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix}.$$

Claim 7. *After running Step 4 and Step 5 in Algorithm 4.3, we have*

$$(4.14) \quad \|\hat{w} - \hat{\zeta}\| \leq L\|\xi - z\|^2 + \frac{1}{1 - \hat{\gamma}_{\mu}(f, z)\|\xi - z\|} \hat{\gamma}_{\mu}(f, z)\|\xi - z\|^2,$$

and

$$(4.15) \quad \|g(w)\| \leq \frac{4\|D\hat{f}(z)\|}{1 - 2\hat{\gamma}_{\mu}(f, z)\|\xi - z\|} \hat{\gamma}_{\mu}(f, z)\|\xi - z\|^2 + lL\|\xi - z\|^2,$$

where

$$(4.16) \quad \zeta \leftarrow W_{\dagger}^* \cdot \xi, \quad g(X) \leftarrow U^* \cdot f(W_{\dagger} \cdot X), \quad w = (w_1, \hat{w}) \leftarrow W_{\dagger}^* y,$$

and l is the Lipschitz constant of the function $f_n(X)$.

Proof. By (4.5) and (4.12), we have

$$\begin{aligned} \|\hat{w} - \hat{\zeta}\| &= \|W_2^*(y_1 - \xi_1) + W_4^*(\hat{y} - \hat{\xi})\| \leq L\|\xi - z\|^2 + \|\hat{y} - \hat{\xi}\| \\ &\leq L\|\xi - z\|^2 + \frac{1}{1 - \hat{\gamma}_{\mu}(f, z)\|\xi - z\|} \hat{\gamma}_{\mu}(f, z)\|\xi - z\|^2. \end{aligned}$$

By (4.6) and (4.13), we have

$$\begin{aligned} \|g(w)\| &= \|U_1^* \hat{f}(y) + U_3^* f_n(y)\| \leq \|\hat{f}(y)\| + L\|\xi - z\| \|f_n(y)\| \\ &\leq \frac{4\|D\hat{f}(z)\|}{1 - 2\hat{\gamma}_{\mu}(f, z)\|\xi - z\|} \hat{\gamma}_{\mu}(f, z)\|\xi - z\|^2 + lL\|\xi - z\|^2. \end{aligned}$$

□

Let Δ_k and Λ_k be differential functionals calculated by (2.7) and (2.10) incrementally from $\Lambda_1 = d_1$ until $\Delta_{\mu}(g_n) = O(1)$. It should be noted that d_1^k is the only differential monomial of the highest order k in Δ_k and no other d_1^s with $s < k$ in Δ_k .

Claim 8. *After running the first six steps in Algorithm 4.3, we have*

$$(4.17) \quad \|x_1 - \zeta_1\| = O(\|\xi - z\|^2),$$

where $x = (x_1, \hat{x}) \leftarrow (N_2(g_n, w), \hat{w})$, $\zeta \leftarrow W_{\dagger}^* \cdot \xi$.

Proof. It is straightforward to check that w is a simple multiple zero of the system

$$\begin{cases} \hat{h}(X) = \hat{g}(X) - \hat{g}(w), \\ h_n(X) = g_n(X) - g_n(w) - \sum_{k=1}^{\mu-1} \Delta_k(g_n)(X_1 - w_1)^k. \end{cases}$$

with multiplicity μ and $Dg(w)$ is of normalized form (2.9) (the construction is very similar to Theorem 8). Thus, by (3.14), we have

$$\begin{aligned} h_n(X) = & - \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} \cdot D\hat{h}(w)^{-1} \underbrace{\hat{h}(X)(X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1}}_{\text{term \#1}} \\ & + \Delta_\mu(h_n)(X_1 - w_1)^\mu + \sum_{i+j=\mu, j>0} C_{i,j} \underbrace{(X_1 - w_1)^i(\hat{X} - \hat{w})^j}_{\text{term \#2}} + \sum_{k \geq \mu+1} \underbrace{\frac{D^k h_n(w)(X - w)^k}{k!}}_{\text{term \#3}} \\ & + \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} \cdot \left(\sum_{k+i+j-1 \geq \mu+1} D\hat{h}(w)^{-1} \underbrace{\frac{D^k \hat{h}(w)(X - w)^k}{k!} (X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1}}_{\text{term \#4}} \right). \end{aligned}$$

Let $g_n(X) = h_n(X) + g_n(w) + \sum_{k=1}^{\mu-1} \Delta_k(g_n)(X_1 - w_1)^k$ and $\{1, \bar{\Lambda}_1, \dots, \bar{\Lambda}_{\mu-1}\}$ be a reduced basis of $\mathcal{D}_{g,\zeta}$,

$$\begin{aligned} 0 = \bar{\Lambda}_{\mu-1}(g_n) &= \bar{\Lambda}_{\mu-1}(h_n) + \Delta_{\mu-1}(g_n) \\ &= \mu \Delta_\mu(h_n)(\zeta_1 - w_1) + \Delta_{\mu-1}(g_n) + O(\|\xi - z\|^2) \\ &= \mu \Delta_\mu(g_n)(\zeta_1 - w_1) + \Delta_{\mu-1}(g_n) + O(\|\xi - z\|^2) \\ &= -\mu \Delta_\mu(g_n)(N_2(w_1) - \zeta_1) + O(\|\xi - z\|^2), \end{aligned}$$

because of the following facts:

- for the term #1, $\bar{\Lambda}_{\mu-1} \left(\hat{h}(X)(X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1} \right) = O(\|\xi - z\|^2)$ for $1 \leq i + j - 1 \leq \mu - 2$, since $\bar{\Lambda}_{\mu-1} \in \mathcal{D}_{g,\zeta}$, $\hat{h}(X) = \hat{g}(X) - \hat{g}(w)$ and $\|\hat{g}(w)\| = O(\|\xi - z\|^2)$ by (4.15);
- for the term #2, $\bar{\Lambda}_{\mu-1} \left((X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1}(\hat{X} - \hat{w}) \right) = O(\|\xi - z\|^2)$, since $j > 0$, $i + j - 1 = \mu - 1$ and $\|\hat{w} - \hat{\zeta}\| = O(\|\xi - z\|^2)$ by (4.14);
- for the term #3, $\bar{\Lambda}_{\mu-1} \left((X - w)^k \right) = O(\|\xi - z\|^2)$ for $k - (\mu - 1) \geq 2$;
- for the term #4, $\bar{\Lambda}_{\mu-1} \left((X - w)^k (X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1} \right) = O(\|\xi - z\|^2)$ for $k + i + j - 1 - (\mu - 1) \geq 2$.

Moreover, we have $\Delta_\mu(g_n) = O(1)$. Therefore, after running Step 6, we have

$$\|x_1 - \zeta_1\| = \|N_2(w_1) - \zeta_1\| = O(\|\xi - z\|^2).$$

□

Now we are ready to prove Theorem 11.

Proof. Both W_\dagger and W_\ddagger are unitary matrices, according to (4.3) and Claim 7, 8, we have

$$\|W_\dagger \cdot W_\ddagger \cdot x - \xi\| = \left\| \begin{pmatrix} x_1 - W_\ddagger^* \cdot W_\dagger^* \cdot \xi_1 \\ \hat{x} - W_\ddagger^* \cdot W_\dagger^* \cdot \hat{\xi} \end{pmatrix} \right\| = O(\|\xi - z\|^2).$$

□

Remark 5. *Theorem 11 can be combined with Theorem 8 to provide an algorithm for computing an certified ball which contains a simple singular solution of f . After running the first six steps in Algorithm 4.3, if the condition (3.23) is satisfied by g at x , then according to Theorem 8, it has μ zeros (counting multiplicities) in the ball of radius $r = \frac{d}{4\gamma_\mu}$ around x , which indicates that the input polynomial system f has μ zeros (counting multiplicities) in the ball $B(N_f(z), r)$.*

The difficulty of giving a quantified quadratic convergence of Step 6 is due to the complicate expression of the polynomial $h_n(X)$. In order to avoid awkward large expressions, in what follows, we only show the proof of the quantitative version of Claim 8 for simple triple zeros. It is clear from proofs given below that there is no significant obstacle to extend the quantified quadratic convergence proof of the Algorithm 4.3 for simple triple zeros to simple multiple zeros of higher multiplicities. This can also be observed by our analysis in Section 3.2, which generalizes results in Section 3.1 for simple tripe zeros to simple multiple zeros of higher multiplicities.

Definition 6. *Let $u = \max\{\gamma_3(f, \xi)^3 \|\xi - z\|, L\gamma_3(f, \xi)^2 \|\xi - z\|\}$. We define the following rational functions:*

$$\begin{aligned}
l_1(u) &= \frac{(1-2u)^2}{(2(1-2u)^2-1) \cdot (1-u)^3}, \\
l_2(u) &= \frac{(2u-1)^6}{(128u^6-384u^5+480u^4-336u^3+140u^2-32u+1) \cdot (1-u)^3}, \\
l_3(u) &= \sqrt{1 + \left(\frac{l_1 u}{1-l_1 u}\right)^2}, \\
b_1(u) &= u + \frac{l_1 u}{1-l_1 u}, \\
b_2(u) &= \frac{(16l_1^2 l_3^2 u^4 - (16l_1^2 l_3^2 + 16l_1 l_3)u^3 + (16l_1 l_3 + 4)u^2 - 4u + 1)^2}{(1-2u)^2(1-2l_1 l_3 u)^2} \\
&\quad \cdot \left[\left(\frac{l_2}{3} + \frac{17l_2 l_3^2}{3}\right)u + \frac{7l_2^2}{3}u^2 + \frac{4l_2^2}{3}u^3 + \left(\frac{l_2}{3} + \frac{7l_2^2 u}{3} + \frac{8l_2^2 u^2}{3}\right) \frac{l_1 u}{1-l_1 u} \right. \\
&\quad + \frac{4l_2^2 u}{3} \left(\frac{l_1 u}{1-l_1 u}\right)^2 + \frac{l_2}{3} \cdot \frac{l_1 l_3^2 u}{1-l_1 l_3 u} \\
&\quad + \frac{8l_2^3 l_3^2 u (12l_2^2 l_3^3 u^3 + (6l_2^2 l_3^2 - 14l_2 l_3^2)u^2 + (4l_3 - 8l_2 l_3)u + 3)}{3(1-2l_2 l_3 u)^3} \\
&\quad \left. + \frac{2l_2 l_1^2 l_3^3 u (16l_1^2 l_3^3 u^3 + (4l_1^2 l_3^2 - 20l_1 l_3)u^2 + (6-6l_1 l_3)u + 3)}{(1-2l_1 l_3 u)^3} \right] \\
&\quad + \frac{l_2^2}{3} \left(u + \frac{l_1 u}{1-l_1 u}\right) \left(\frac{2(-4l_1^2 l_3^2 u^2 + 4l_1 l_3 u)^2}{(1-2u)^4(1-2l_1 l_3 u)^4} + \frac{7(-4l_1^2 l_3^2 u^2 + 4l_1 l_3 u)}{(1-2u)^2(1-2l_1 l_3 u)^2} + 8\right).
\end{aligned}$$

Theorem 12. *Given an approximate zero z of a system f associated to a simple triple zero ξ of multiplicity 3 and satisfying $f(\xi) = 0$, $\dim \ker Df(\xi) = 1$.*

(1) *If $u < u_3 \approx 0.0137$, where u_3 is the smallest positive solution of the equation:*

$$b_1(u)^2 + b_2(u)^2 = 1,$$

then the output of Algorithm 4.3 satisfies:

$$\|N_f(z) - \xi\| < \|z - \xi\|.$$

(2) If $u < u'_3 \approx 0.0098$, where u'_3 is the smallest positive solution of the equation:

$$b_1(u)^2 + b_2(u)^2 = \frac{1}{4},$$

then after k times of iteration we have

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

The proof of Theorem 12 is based on the following facts.

Claim 9. When $u \leq u_3$, we have

$$(4.18) \quad \hat{\gamma}_3(f, z) \leq l_1 \cdot \gamma_3(f, \xi),$$

$$(4.19) \quad \gamma_{3,n}(f, z) \leq l_2 \cdot \gamma_3(f, \xi).$$

Proof. For $k \geq 2$, by the Taylor expansion of $D^k \hat{f}(z)$ at ξ , we have

$$\begin{aligned} & \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \\ & \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left\| D\hat{f}(\xi)^{-1} \left(\frac{D^k \hat{f}(\xi)}{k!} + \sum_{i \geq 1} \frac{D^{k+i} \hat{f}(\xi)}{k!i!} (\xi - z)^i \right) \right\| \\ & \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\gamma_3(f, \xi)^{k-1} + \sum_{i \geq 1} \frac{(k+i)!}{k!i!} \gamma_3(f, \xi)^{k+i-1} \|\xi - z\|^i \right) \\ & \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \frac{\gamma_3(f, \xi)^{k-1}}{(1 - \gamma_3(f, \xi) \|\xi - z\|)^{k+1}} \\ & \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\frac{1}{1-u} \right)^{k+1} \gamma_3(f, \xi)^{k-1}. \end{aligned}$$

Then by the definition of $\hat{\gamma}_3$ and the above inequalities, we have

$$\begin{aligned} (4.20) \quad \hat{\gamma}_3(f, z) & \leq \max_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\|^{\frac{1}{k-1}} \\ & \leq \max_{k \geq 2} \left(\left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\|^{\frac{1}{k-1}} \cdot \left(\frac{1}{1-u} \right)^{\frac{k+1}{k-1}} \gamma_3(f, \xi) \right) \\ & \leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left(\frac{1}{1-u} \right)^3 \gamma_3(f, \xi). \end{aligned}$$

According to Remark 4, for $u \leq u_3 \approx 0.0137$, we can show that:

$$\left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \leq \frac{(1-2u)^2}{2(1-2u)^2-1},$$

therefore (4.18) holds by (4.20) and Definition 6.

Similar to the proof of inequality (4.20), we have

$$(4.21) \quad \gamma_{3,n}(f, z) \leq \|\Delta_3(f_n)(z)^{-1} \Delta_3(f_n)(\xi)\| \cdot \left(\frac{1}{1-u}\right)^3 \gamma_3(f, \xi).$$

To prove (4.19), we notice that:

$$\Delta_3(f_n)(z)^{-1} \Delta_3(f_n)(\xi) = (1 + (\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1))^{-1}.$$

By the Taylor expansion of $\frac{\partial^3 f(z)}{\partial X_1^3}$ and $\frac{\partial^2 f(z)}{\partial X_1 \partial \hat{X}}$ at ξ , we have:

$$\begin{aligned} \Delta_3(f_n)(z) &= \frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_1^3} + a_{2,z} \cdot \frac{\partial^2 f_n(z)}{\partial X_1 \partial \hat{X}} \\ &= \frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^3} + \sum_{k \geq 1} \sum_{i=0}^k \frac{1}{6 \cdot i!(k-i)!} \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{3+i} \partial \hat{X}^{k-i}} (\xi_1 - z)^i (\hat{\xi} - \hat{z})^{k-i} \\ &\quad + a_{2,z} \cdot \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial \hat{X}} + a_{2,z} \cdot \sum_{k \geq 1} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^{k+2} f_n(\xi)}{\partial X_1^{1+i} \partial \hat{X}^{1+k-i}} (\xi_1 - z)^i (\hat{\xi} - \hat{z})^{k-i} \\ &= \Delta_3(f_n)(\xi) + \sum_{k \geq 1} \sum_{i=0}^k \frac{1}{6 \cdot i!(k-i)!} \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{3+i} \partial \hat{X}^{k-i}} (\xi_1 - z)^i (\hat{\xi} - \hat{z})^{k-i} \\ &\quad + a_{2,z} \cdot \sum_{k \geq 1} \sum_{i=0}^k \frac{1}{i!(k-i)!} \frac{\partial^{k+2} f_n(\xi)}{\partial X_1^{1+i} \partial \hat{X}^{1+k-i}} (\xi_1 - z)^i (\hat{\xi} - \hat{z})^{k-i}, \end{aligned}$$

where

$$a_{2,z} = D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2}.$$

Therefore, we have

$$\begin{aligned} (4.22) \quad &\|\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1\| \\ &\leq \sum_{k \geq 1} \sum_{i=0}^k \frac{(k+3)!}{6 \cdot i!(k-i)!} \gamma_{3,n}(f, \xi)^{k+2} \|\xi - z\|^k \\ &\quad + \|a_{2,z}\| \cdot \sum_{k \geq 1} \sum_{i=0}^k \frac{(k+2)!}{i!(k-i)!} \gamma_{3,n}(f, \xi)^{k+1} \|\xi - z\|^k \\ &\leq \frac{1}{6} \sum_{k \geq 1} (k+3)(k+2)(k+1) 2^k \gamma_{3,n}(f, \xi)^{k+2} \|\xi - z\|^k \\ &\quad + \|a_{2,z}\| \cdot \sum_{k \geq 1} (k+2)(k+1) 2^k \gamma_{3,n}(f, \xi)^{k+1} \|\xi - z\|^k, \end{aligned}$$

where

$$\begin{aligned} \|a_{2,z}\| &= \left\| D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| \\ &\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \\ &\quad \left\| D\hat{f}(\xi)^{-1} \left(\frac{1}{2} \frac{\partial^2 \hat{f}(\xi)}{\partial X_1^2} + \sum_{k \geq 1} \sum_{i=0}^k \frac{1}{2 \cdot i!(k-i)!} \frac{\partial^{k+2} f(\xi)}{\partial X_1^{2+i} \partial \hat{X}^{k-i}} (\xi_1 - z)^i (\hat{\xi} - \hat{z})^{k-i} \right) \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \hat{\gamma}_3(f, \xi) + \sum_{k \geq 1} \sum_{i=0}^k \frac{(k+2)!}{2 \cdot i! (k-i)!} \hat{\gamma}_3(f, \xi)^{k+1} \|\xi - z\|^k \\
&\leq \hat{\gamma}_3(f, \xi) + \frac{1}{2} \sum_{k \geq 1} (k+2)(k+1) 2^k \hat{\gamma}_3(f, \xi)^{k+1} \|\xi - z\|^k \\
&\leq \hat{\gamma}_3(f, \xi) + \frac{-2u(4u^2 - 6u + 3)}{(2u-1)^3}.
\end{aligned}$$

By (4.22), we have:

$$\begin{aligned}
&\|\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1\| \\
&\leq \frac{-8u(2u^3 - 4u^2 + 3u - 1)}{(2u-1)^4} + \frac{-2u(4u^2 - 6u + 3)}{(2u-1)^3} \cdot \frac{-4u(4u^2 - 6u + 3)}{(2u-1)^3} \\
&\quad + \sum_{k \geq 1} (k+2)(k+1) 2^k \gamma_{3,n}(f, \xi)^{k+2} \|\xi - z\|^k \\
&\leq \frac{-4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5)}{(2u-1)^6}.
\end{aligned}$$

When $u \leq u_3$,

$$\frac{-4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5)}{(2u-1)^6} < 1,$$

then we have

$$\begin{aligned}
\|\Delta_3(f_n)(z)^{-1} \Delta_3(f_n)(\xi)\| &= \|(1 + (\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1))^{-1}\| \\
&\leq \frac{1}{1 - \frac{-4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5)}{(2u-1)^6}} \\
&= \frac{(2u-1)^6}{(128u^6 - 384u^5 + 480u^4 - 336u^3 + 140u^2 - 32u + 1)}.
\end{aligned}$$

Hence, the inequality (4.19) holds. \square

We notice that after running first five steps in Algorithm 4.3, we have

$$\zeta \leftarrow W_{\dagger}^* \cdot \xi, \quad g(X) \leftarrow U^* \cdot f(W_{\dagger} \cdot X), \quad w \leftarrow W_{\dagger}^* y, \quad y \leftarrow (z_1, N_1(\hat{f}, \hat{z})).$$

For unitary matrices U and W_{\dagger} , we have $\gamma_3(f, \xi) = \gamma_3(U^* \cdot f, W_{\dagger}^* \cdot \xi)$ and the following inequalities hold:

$$\begin{aligned}
\hat{\gamma}_3(g, w) &\leq l_1 \cdot \gamma_3(g, \zeta) = l_1 \cdot \gamma_3(f, \xi), \\
\gamma_{3,n}(g, w) &\leq l_2 \cdot \gamma_3(g, \zeta) = l_2 \cdot \gamma_3(f, \xi).
\end{aligned}$$

Claim 10. *When $u \leq u_3$, we have*

$$(4.23) \quad \|w - \zeta\| \leq l_3 \cdot \|z - \xi\|.$$

Proof. It is clear that

$$\|w - \zeta\| = \|W_{\dagger}^*(y - \xi)\| = \|y - \xi\|.$$

By Claim 6 and Claim 9, when $u \leq u_3$, we have:

$$\|\hat{y} - \hat{\xi}\| \leq \frac{1}{1 - \hat{\gamma}_3(f, z) \|\xi - z\|} \hat{\gamma}_3(f, z) \|\xi - z\|^2$$

$$\leq \frac{l_1 u}{1 - l_1 u} \|\xi - z\|.$$

By the fact that $y_1 = z_1$, when $u \leq u_3$, we have

$$\|w - \zeta\| = \|y - \xi\| \leq \sqrt{1 + \left(\frac{l_1 u}{1 - l_1 u}\right)^2} \|\xi - z\| = l_3 \cdot \|z - \xi\|.$$

□

In Claim 8, we have shown that

$$\|x_1 - \zeta_1\| = O(\|\xi - z\|^2).$$

Below, we give a quantitative version of Claim 8 for the simple triple zero case.

Following the proof of Claim 8, we consider the following system:

$$\begin{cases} \hat{h}(X) = \hat{g}(X) - \hat{g}(w) \\ h_n(X) = g_n(X) - g_n(w) - \sum_{k=1}^2 \Delta_k(g_n)(X_1 - w_1)^k \end{cases}$$

w is a simple triple zero of $h(X)$ whose Jacobian matrix is of normalized form. The polynomial $h_n(X)$ can be written as:

$$\begin{aligned} h_n(X) &= - \sum_{i+j=1} T_{i,j} \cdot D\hat{h}(w)^{-1} \hat{h}(X) (X_1 - w_1)^i (\hat{X} - \hat{w})^j \\ &\quad + \Delta_3(h_n)(X_1 - w_1)^3 + \sum_{i+j=3, j>0} C_{i,j} (X_1 - w_1)^i (\hat{X} - \hat{w})^j \\ &\quad + \sum_{k \geq 4} \frac{D^k h_n(w)(X - w)^k}{k!} \\ &\quad + \sum_{i+j=1} T_{i,j} \cdot \sum_{k \geq 3} D\hat{h}(w)^{-1} \frac{D^k \hat{h}(w)(X - w)^k}{k!} (X_1 - w_1)^i (\hat{X} - \hat{w})^j \\ &\triangleq \Delta_3(h_n)(X_1 - w_1)^3 + B. \end{aligned}$$

It is clear that $D\hat{h}(w) = D\hat{g}(w)$, $D^k h(w) = D^k g(w)$ for $k \geq 3$,

$$\begin{aligned} C_{2,1} &= \frac{1}{2} \frac{\partial^3 g_n(w)}{\partial X_1^2 \partial \hat{X}} - \frac{\partial^2 g_n(w)}{\partial X_1 \partial \hat{X}} \cdot D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X_1 \partial \hat{X}} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial \hat{X}^2} \cdot D\hat{g}(w)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(w)}{\partial X_1^2}, \\ C_{1,2} &= \frac{1}{2} \frac{\partial^3 g_n(w)}{\partial X_1 \partial \hat{X}^2} - \frac{\partial^2 g_n(w)}{\partial X_1 \partial \hat{X}} \cdot D\hat{g}(w)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(w)}{\partial \hat{X}^2} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial \hat{X}^2} \cdot D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X_1 \partial \hat{X}}, \\ C_{0,3} &= \frac{1}{6} \frac{\partial^3 g_n(w)}{\partial \hat{X}^3} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial \hat{X}^2} \cdot D\hat{g}(w)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(w)}{\partial \hat{X}^2}, \\ T_{1,0} &= - \frac{\partial^2 g_n(w)}{\partial X_1 \partial \hat{X}}, \\ T_{0,1} &= - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial \hat{X}^2}. \end{aligned}$$

By the definition of $\hat{\gamma}_3(g, w)$ and $\gamma_{3,n}$, we also have the following facts:

$$\begin{aligned} \|\Delta_3(g_n)^{-1} C_{2,1}\| &\leq 3\gamma_{3,n}(g, w)^2 + 2\gamma_{3,n}(g, w) \cdot 2\hat{\gamma}_3(g, w) + \gamma_{3,n}(g, w) \cdot \hat{\gamma}_3(g, w) \\ &\leq 8\gamma_3(g, w)^2, \end{aligned}$$

$$\begin{aligned}
\|\Delta_3(g_n)^{-1}C_{1,2}\| &\leq 3\gamma_{3,n}(g,w)^2 + 2\gamma_3(g,w) \cdot \hat{\gamma}_3(g,w) + \gamma_{3,n}(g,w) \cdot 2\hat{\gamma}_3(g,w) \\
&\leq 7\gamma_3(g,w)^2, \\
\|\Delta_3(g_n)^{-1}C_{0,3}\| &\leq \gamma_{3,n}(g,w)^2 + \gamma_{3,n}(g,w) \cdot \hat{\gamma}_3(g,w) \\
&\leq 2\gamma_3(g,w)^2, \\
\|\Delta_3(g_n)^{-1}T_{1,0}\| &\leq 2\gamma_{3,n}(g,w), \\
\|\Delta_3(g_n)^{-1}T_{0,1}\| &\leq \gamma_{3,n}(g,w).
\end{aligned}$$

Let $\{1, \bar{\Lambda}_1, \bar{\Lambda}_2\}$ be the reduced basis of $\mathcal{D}_{g,\zeta}$ and $\bar{\Lambda}_1 = d_1 + a_1 d_2$, then

$$\bar{\Delta}_2 = d_1^2 + a_1 d_1 d_2 + a_1^2 d_2^2,$$

$$\bar{\Lambda}_2 = \bar{\Delta}_2 + a_2 d_2,$$

where $a_1 = D\hat{g}(\zeta)^{-1} \frac{\partial \hat{g}(\zeta)}{\partial X_1}$, $a_2 = D\hat{g}(\zeta)^{-1} \bar{\Delta}_2(\hat{g})(\zeta)$.

By Taylor expansion of $\frac{\partial \hat{g}(\zeta)}{\partial X_1}$ at w

$$\frac{\partial \hat{g}(\zeta)}{\partial X_1} = \sum_{k \geq 2} \sum_{i=1}^k \frac{1}{(i-1)!(k-i)!} \frac{\partial^k \hat{g}(w)}{\partial X_1^i \partial \hat{X}^{k-i}} (\zeta_1 - w_1)^{i-1} (\hat{\zeta} - \hat{w})^{k-i},$$

and $\gamma_3(g, \zeta) = \gamma_3(f, \xi)$, we have

$$\begin{aligned}
\|a_1\| &= \left\| D\hat{g}(\zeta)^{-1} \frac{\partial \hat{g}(\zeta)}{\partial X_1} \right\| \\
&\leq \|D\hat{g}(\zeta)^{-1} D\hat{g}(w)\| \cdot \left\| D\hat{g}(w)^{-1} \frac{\partial \hat{g}(\zeta)}{\partial X_1} \right\| \\
&\leq \|D\hat{g}(\zeta)^{-1} D\hat{g}(w)\| \cdot \sum_{k \geq 2} \sum_{i=1}^k \frac{k!}{(i-1)!(k-i)!} \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-1} \\
&= \|D\hat{g}(\zeta)^{-1} D\hat{g}(w)\| \cdot \frac{4l_1 \gamma_3(f, \xi) l_3 \|\xi - z\| (1 - l_1 \gamma_3(f, \xi) l_3 \|\xi - z\|)}{(1 - 2l_1 \gamma_3(f, \xi) l_3 \|\xi - z\|)^2} \\
&\leq \frac{1}{(1-2u)^2} \cdot \frac{4l_1 l_3 u (1 - l_1 l_3 u)}{(1 - 2l_1 l_3 u)^2} \\
&= \frac{4l_1 l_3 u (1 - l_1 l_3 u)}{(1-2u)^2 (1 - 2l_1 l_3 u)^2},
\end{aligned}$$

and

$$\begin{aligned}
\|a_2\| &= \|D\hat{g}(\zeta)^{-1} \bar{\Delta}_2(\hat{g})(\zeta)\| \\
&\leq \left\| D\hat{g}(\zeta)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(\zeta)}{\partial X_1^2} \right\| + \|a_1\| \cdot \left\| D\hat{g}(\zeta)^{-1} \frac{\partial^2 \hat{g}(\zeta)}{\partial X_1 \partial \hat{X}} \right\| \\
&\quad + \|a_1\|^2 \cdot \left\| D\hat{g}(\zeta)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(\zeta)}{\partial \hat{X}^2} \right\| \\
&\leq \gamma_3(g, \zeta) + 2\|a_1\| \gamma_3(g, \zeta) + \|a_1\|^2 \gamma_3(g, \zeta).
\end{aligned}$$

Claim 11. *We have*

$$N_2(g_n, w) - \zeta_1 = \frac{1}{3} \Delta_3(g_n)^{-1} \bar{\Lambda}_2(B).$$

Proof. As $\bar{\Lambda}_2 = \bar{\Delta}_2 + a_2 d_2$, applying $\bar{\Lambda}_2$ on both sides of the equation:

$$g_n(X) = h_n(X) + g_n(w) + \sum_{k=1}^2 \Delta_k(g_n)(X_1 - w_1)^k,$$

we have:

$$\begin{aligned} 0 &= \bar{\Lambda}_2(g_n) = \bar{\Lambda}_2(h_n) + \Delta_2(g_n) \\ &= 3\Delta_3(h_n)(\zeta_1 - w_1) + \Delta_2(g_n) + \bar{\Lambda}_2(B) \\ &= 3\Delta_3(g_n)(\zeta_1 - w_1) + \Delta_2(g_n) + \bar{\Lambda}_2(B). \end{aligned}$$

Therefore, we have

$$N_2(g_n, w) - \zeta_1 = w_1 - \frac{1}{3}\Delta_3(g_n)^{-1}\Delta_2(g_n) - \zeta_1 = \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2(B).$$

□

Claim 12. When $u \leq u_3$, we have

$$(4.24) \quad |x_1 - \zeta_1| \leq b_2(u)\|z - \xi\|.$$

Proof. By Claim 11 and above arguments, we have

$$\begin{aligned} |N_2(g_n, w) - \zeta_1| &= \left| \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2(B) \right| \\ &= \left| \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2 \left(T_{1,0} \cdot D\hat{h}(w)^{-1}\hat{h}(X)(X_1 - w_1) + T_{0,1} \cdot D\hat{h}(w)^{-1}\hat{h}(X)(\hat{X} - \hat{w}) \right) \right| \\ &\quad + \left| \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2 \left(\sum_{i+j=3, j>0} C_{i,j}(X_1 - w_1)^i(\hat{X} - \hat{w})^j \right) \right| \\ &\quad + \left| \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2 \left(\sum_{k \geq 4} \frac{D^k h_n(w)(X - w)^k}{k!} \right) \right| \\ &\quad + \left| \frac{1}{3}\Delta_3(g_n)^{-1}\bar{\Lambda}_2 \left(\sum_{i+j=1} T_{i,j} \cdot \sum_{k \geq 3} D\hat{h}(w)^{-1} \frac{D^k \hat{h}(w)(X - w)^k}{k!} (X_1 - w_1)^i(\hat{X} - \hat{w})^j \right) \right| \\ &= \frac{1}{3} \left\| \Delta_3(g_n)^{-1}T_{0,1} \right\| \cdot \left\| D\hat{h}(w)^{-1}\hat{h}(\zeta) \right\| \|a_2\| \\ &\quad + \frac{1}{3} \left\| \Delta_3(g_n)^{-1} \cdot \left(C_{2,1} + C_{1,2}(a_1 + a_2(\zeta_1 - w_1)) + C_{0,3}(a_1^2 + 2a_2(\hat{\zeta} - \hat{w})) \right) \right\| \|\hat{\zeta} - \hat{w}\| \\ &\quad + \frac{1}{3} \cdot \sum_{i+j=3, j>0} \left\| \Delta_3(g_n)^{-1}C_{i,j} \right\| |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^{j-1} \|a_2\| \\ &\quad + \frac{1}{3} \cdot \frac{1}{2} \sum_{k \geq 4} \sum_{i=2}^k \frac{i(i-1)}{i!(k-i)!} \left\| \Delta_3(g_n)^{-1} \frac{\partial^k h_n(w)}{\partial X_1^i \hat{X}^{k-i}} \right\| |\zeta_1 - w_1|^{i-2} \|\hat{\zeta} - \hat{w}\|^{k-i} \\ &\quad + \frac{1}{3} \cdot \|a_1\| \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{i(k-i)}{i!(k-i)!} \left\| \Delta_3(g_n)^{-1} \frac{\partial^k h_n(w)}{\partial X_1^i \hat{X}^{k-i}} \right\| |\zeta_1 - w_1|^{i-1} \|\hat{\zeta} - \hat{w}\|^{k-i-1} \\ &\quad + \frac{1}{3} \cdot \frac{1}{2} \|a_1\|^2 \sum_{k \geq 4} \sum_{i=0}^{k-2} \frac{(k-i)(k-i-1)}{i!(k-i)!} \left\| \Delta_3(g_n)^{-1} \frac{\partial^k h_n(w)}{\partial X_1^i \hat{X}^{k-i}} \right\| |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^{k-i-2} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3} \cdot \|a_2\| \sum_{k \geq 4} \sum_{i=0}^{k-1} \frac{(k-i)}{i!(k-i)!} \left\| \Delta_3(g_n)^{-1} \frac{\partial^k h_n(w)}{\partial X_1^i \hat{X}^{k-i}} \right\| |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^{k-i-1} \\
& + \frac{1}{3} \sum_{i+j=1} \| \Delta_3(g_n)^{-1} T_{i,j} \| \cdot \sum_{k \geq 3} \left(\frac{1}{2} \sum_{l=2}^k \frac{l(l-1)}{l!(k-l)!} \left\| D\hat{h}(w)^{-1} \frac{\partial^k \hat{h}(w)}{\partial X_1^l \hat{X}^{k-l}} \right\| |\zeta_1 - w_1|^{l-2} \|\hat{\zeta} - \hat{w}\|^{k-l} \right. \\
& + \|a_1\| \sum_{l=1}^{k-1} \frac{l(k-l)}{l!(k-l)!} \left\| D\hat{h}(w)^{-1} \frac{\partial^k \hat{h}(w)}{\partial X_1^l \hat{X}^{k-l}} \right\| |\zeta_1 - w_1|^{l-1} \|\hat{\zeta} - \hat{w}\|^{k-l-1} \\
& + \frac{1}{2} \|a_1\|^2 \sum_{l=0}^{k-2} \frac{(k-l)(k-l-1)}{l!(k-l)!} \left\| D\hat{h}(w)^{-1} \frac{\partial^k \hat{h}(w)}{\partial X_1^l \hat{X}^{k-l}} \right\| |\zeta_1 - w_1|^l \|\hat{\zeta} - \hat{w}\|^{k-l-2} \\
& \left. + \|a_2\| \sum_{l=0}^{k-1} \frac{(k-l)}{l!(k-l)!} \left\| D\hat{h}(w)^{-1} \frac{\partial^k \hat{h}(w)}{\partial X_1^l \hat{X}^{k-l}} \right\| |\zeta_1 - w_1|^l \|\hat{\zeta} - \hat{w}\|^{k-l-1} \right) |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^j \\
& \leq \frac{1}{3} \gamma_3(g, w) \left(\|\hat{\zeta} - \hat{w}\| + \frac{1}{1 - \hat{\gamma}_3(g, w) \|\zeta - w\|} \hat{\gamma}_3(g, w) \|\zeta - w\|^2 \right) \|a_2\| \\
& + \frac{1}{3} (8\gamma_3(g, w)^2 + 7\gamma_3(g, w)^2 \|a_1\| + 2\gamma_3(g, w)^2 \|a_1\|^2) \|\hat{\zeta} - \hat{w}\| \\
& + \frac{7}{3} \gamma_3(g, w)^2 \|a_2\| |\zeta_1 - w_1| \|\hat{\zeta} - \hat{w}\| + \frac{4}{3} \gamma_3(g, w)^2 \|a_2\| \|\hat{\zeta} - \hat{w}\|^2 \\
& + \frac{17}{3} \gamma_3(g, w) \|\zeta - w\|^2 \|a_2\| \\
& + \frac{1}{6} \sum_{k \geq 4} \sum_{i=2}^k \frac{i(i-1)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} |\zeta_1 - w_1|^{i-2} \|\hat{\zeta} - \hat{w}\|^{k-i} \\
& + \frac{1}{3} \|a_1\| \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{i(k-i)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} |\zeta_1 - w_1|^{i-1} \|\hat{\zeta} - \hat{w}\|^{k-i-1} \\
& + \frac{1}{6} \|a_1\|^2 \sum_{k \geq 4} \sum_{i=0}^{k-2} \frac{(k-i)(k-i-1)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^{k-i-2} \\
& + \frac{1}{3} \|a_2\| \sum_{k \geq 4} \sum_{i=0}^{k-1} \frac{(k-i)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} |\zeta_1 - w_1|^i \|\hat{\zeta} - \hat{w}\|^{k-i-1} \\
& + \gamma_3(g, w) \cdot \sum_{k \geq 3} \left(\frac{1}{2} \sum_{l=2}^k \frac{l(l-1)k!}{l!(k-l)!} \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \right. \\
& + \|a_1\| \sum_{l=1}^{k-1} \frac{l(k-l)k!}{l!(k-l)!} \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \\
& + \frac{1}{2} \|a_1\|^2 \sum_{l=0}^{k-2} \frac{(k-l)(k-l-1)k!}{l!(k-l)!} \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \\
& \left. + \|a_2\| \sum_{l=0}^{k-1} \frac{(k-l)k!}{l!(k-l)!} \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-1} \right) \|\zeta - w\|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \gamma_3(g, w) \left(\|\hat{\zeta} - \hat{w}\| + \frac{1}{1 - \hat{\gamma}_3(g, w) \|\zeta - w\|} \hat{\gamma}_3(g, w) \|\zeta - w\|^2 \right) \|a_2\| \\
&\quad + \frac{1}{3} (8\gamma_3(g, w)^2 + 7\gamma_3(g, w)^2 \|a_1\| + 2\gamma_3(g, w)^2 \|a_1\|^2) \|\hat{\zeta} - \hat{w}\| \\
&\quad + \frac{7}{3} \gamma_3(g, w)^2 \|a_2\| \|\zeta_1 - w_1\| \|\hat{\zeta} - \hat{w}\| + \frac{4}{3} \gamma_3(g, w)^2 \|a_2\| \|\hat{\zeta} - \hat{w}\|^2 \\
&\quad + \frac{17}{3} \gamma_3(g, w) \|\zeta - w\|^2 \|a_2\| \\
&\quad + \frac{1}{6} \sum_{k \geq 4} 2^{k-2} k(k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \\
&\quad + \frac{1}{3} \|a_1\| \sum_{k \geq 4} 2^{k-2} k(k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \\
&\quad + \frac{1}{6} \|a_1\|^2 \sum_{k \geq 4} 2^{k-2} k(k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2} \\
&\quad + \frac{1}{3} \|a_2\| \sum_{k \geq 4} 2^{k-1} k \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-1} \\
&\quad + \frac{1}{2} \gamma_3(g, w) \sum_{k \geq 3} (1 + 2\|a_1\| + \|a_1\|^2) 2^{k-2} k(k-1) \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^{k-1} \\
&\quad + \gamma_3(g, w) \sum_{k \geq 3} \|a_2\| 2^{k-1} k \hat{\gamma}_3(g, w)^{k-1} \|\zeta - w\|^k \\
&\leq \frac{l_2}{3} (1 + 2\|a_1\| + \|a_1\|^2) \left(u + \frac{l_1 u}{1 - l_1 u} \right) \|\xi - z\| \\
&\quad + \frac{l_2}{3} (1 + 2\|a_1\| + \|a_1\|^2) \cdot \frac{l_1 l_3^2 u}{1 - l_1 l_3 u} \|\xi - z\| \\
&\quad + \frac{l_2^2}{3} (8 + 7\|a_1\| + 2\|a_1\|^2) \left(u + \frac{l_1 u}{1 - l_1 u} \right) \|\xi - z\| \\
&\quad + \frac{7l_2^2}{3} (1 + 2\|a_1\| + \|a_1\|^2) \left(u^2 + \frac{l_1 u^2}{1 - l_1 u} \right) \|\xi - z\| \\
&\quad + \frac{4l_2^2}{3} (1 + 2\|a_1\| + \|a_1\|^2) u \left(u + \frac{l_1 u}{1 - l_1 u} \right)^2 \|\xi - z\| \\
&\quad + \frac{17l_2}{3} (1 + 2\|a_1\| + \|a_1\|^2) l_3^2 u \|\xi - z\| \\
&\quad + \frac{1}{6} (1 + 2\|a_1\| + \|a_1\|^2) \cdot \frac{8l_2^3 l_3^2 u (12l_2^3 l_3^2 u^2 - 16l_2 l_3 u + 6)}{(1 - 2l_2 l_3 u)^3} \|\xi - z\| \\
&\quad + \frac{1}{3} (1 + 2\|a_1\| + \|a_1\|^2) u \cdot \frac{8l_2^3 l_3^3 u (4 - 6l_2 l_3 u)}{(1 - 2l_2 l_3 u)^2} \|\xi - z\| \\
&\quad + \frac{l_2}{2} (1 + 2\|a_1\| + \|a_1\|^2) \cdot \frac{4l_1^2 l_3^2 u (4l_1^2 l_3^2 u^2 - 6l_1 l_3 u + 3)}{(1 - 2l_1 l_3 u)^3} \|\xi - z\| \\
&\quad + (1 + 2\|a_1\| + \|a_1\|^2) l_2 u \cdot \frac{4l_1^2 l_3^3 u (3 - 4l_1 l_3 u)}{(1 - 2l_1 l_3 u)^2} \|\xi - z\| \\
&= b_2(u) \|z - \xi\|.
\end{aligned}$$

□

When $u < u_3 \approx 0.0137$, by Claim 7, 12 and Definition 6, we have:

$$\begin{aligned} \|\hat{x} - \hat{\zeta}\| &\leq L\|\xi - z\|^2 + \frac{1}{1 - \hat{\gamma}_\mu(f, z)\|\xi - z\|} \hat{\gamma}_3(f, z)\|\xi - z\|^2 \\ &\leq \left(u + \frac{l_1 u}{1 - l_1 u}\right) \|z - \xi\| \\ &= b_1(u) \|z - \xi\|. \end{aligned}$$

Now we can complete the proof for Theorem 12.

Proof. (1) When $u < u_3 \approx 0.0137$, $b_1(u)^2 + b_2(u)^2 < 1$, by Claim 12, we have

$$\begin{aligned} \|N_f(z) - \xi\| &= \|W_{\dagger} \cdot W_{\ddagger} \cdot (x - \zeta)\| \\ &= \|x - \zeta\| \\ &= \sqrt{|x_1 - \zeta_1|^2 + \|\hat{x} - \hat{\zeta}\|^2} \\ &\leq \sqrt{b_1(u)^2 + b_2(u)^2} \|z - \xi\| \\ &< \|z - \xi\|. \end{aligned}$$

(2) When $u < u'_3 \approx 0.0098$, $b_1(u)^2 + b_2(u)^2 < \frac{1}{4}$, by Claim 12, we have

$$\begin{aligned} \|N_f(z) - \xi\| &= \|W_{\dagger} \cdot W_{\ddagger} \cdot (x - \zeta)\| \\ &= \|x - \zeta\| \\ &= \sqrt{|x_1 - \zeta_1|^2 + \|\hat{x} - \hat{\zeta}\|^2} \\ &\leq \sqrt{b_1(u)^2 + b_2(u)^2} \|z - \xi\| \\ &< \frac{1}{2} \|z - \xi\|. \end{aligned}$$

Hence, the following inequality is true for $k = 1$:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k - 1} \|z - \xi\|.$$

For $k \geq 2$, assume by induction that

$$\|N_f^{k-1}(z) - \xi\| < \left(\frac{1}{2}\right)^{2^{k-1} - 1} \|z - \xi\|.$$

Let $u^{(k-1)} = \gamma_3(f, \xi)^3 \|N_f^{k-1}(z) - \xi\|$. For $0 < u < u'_3$, we have $u^{(k-1)} < u$ and $\frac{\sqrt{b_1(u)^2 + b_2(u)^2}}{u}$ is increasing. Therefore, by induction we have

$$\begin{aligned} &\|N_f^k(z) - \xi\| \\ &< \frac{\sqrt{b_1(u^{(k-1)})^2 + b_2(u^{(k-1)})^2} \gamma_3(f, \xi)^3}{u^{(k-1)}} \|N_f^{k-1}(z) - \xi\|^2 \\ &< \frac{\sqrt{b_1(u)^2 + b_2(u)^2} \gamma_3(f, \xi)^3}{u} \|N_f^{k-1}(z) - \xi\|^2 \\ &< \frac{\sqrt{b_1(u)^2 + b_2(u)^2} \gamma_3(f, \xi)^3}{u} \left(\frac{1}{2}\right)^{2^k - 2} \|z - \xi\|^2 \end{aligned}$$

$$\leq \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

□

Remark 6. The unitary transformations in Algorithm 4.3 may convert a sparse polynomial system into a dense polynomial system, therefore, the computations of the modified Newton iterations may become more costly. Hence, in our implementation, we use the chain rule to avoid storing or computing with dense polynomial systems obtained after performing unitary transformations. For example, suppose $g(X) = U^* \cdot f(W_{\dagger} \cdot X)$, then we have

$$(4.25) \quad Dg(X) = U^* \cdot Df(W_{\dagger} \cdot X) \cdot W_{\dagger}.$$

Let $y = W_{\dagger}^* z$, we have

$$(4.26) \quad Dg(y) = U^* \cdot Df(z) \cdot W_{\dagger}.$$

Instead of evaluating $Dg(X)$ at y , we evaluate $Df(X)$ at z and then perform matrix multiplications, which avoids storing and computing the dense system $Dg(X)$.

Similarly, as we have already demonstrated in [25, Example 3.1], instead of computing and storing dense differential functionals Δ_k and Λ_k , we compute polynomials $L_k(g)$ and $P_k(g)$ as

$$(4.27) \quad P_k(g) = \sum_{j=1}^{k-1} \frac{j}{k} \cdot D(L_{k-j}(g)) \cdot \mathbf{a}_j \text{ and } L_k(g) = P_k(g) + Dg \cdot \mathbf{a}_k,$$

where L_k and P_k are differential operators corresponding to Δ_k and Λ_k respectively, $\mathbf{a}_1 = (1, 0, \dots, 0)^T$ and $\mathbf{a}_k = (0, a_{k,2}, \dots, a_{k,n})^T$. The polynomial system $P_k(g)$ and the value $\Delta_k(g)$ can be obtained by applying the chain rule (4.25) and (4.26) recursively to (4.27).

Remark 7. The Maple codes of three Algorithms and test results are available via request.

Although the algorithms and proofs of quadratic convergence given in the paper are for polynomial systems with exact simple multiple zeros, examples are given to demonstrate that our algorithms are also applicable to analytic systems and polynomial systems with a cluster of simple roots.

REFERENCES

- [1] Carlos A. Berenstein, Alekos Vidras, Roger Gay, and Alain Yger. *Residue currents and Bézout identities*. Birkhauser, 1993.
- [2] Lenore Blum, Felipe Cucker, Michael Shub, and Steve Smale. *Complexity and Real Computation*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1998.
- [3] B. Dayton, T. Li, and Z. Zeng. Multiple zeros of nonlinear systems. *Mathematics of Computation*, 80:2143–2168, 2011.
- [4] B. Dayton and Z. Zeng. Computing the multiplicity structure in solving polynomial systems. In M. Kauers, editor, *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation*, pages 116–123, New York, NY, USA, 2005. ACM.
- [5] D. W. Decker and C. T. Kelley. Newton’s method at singular points I. *SIAM Journal on Numerical Analysis*, 17:66–70, 1980.
- [6] D. W. Decker and C. T. Kelley. Newton’s method at singular points II. *SIAM Journal on Numerical Analysis*, 17:465–471, 1980.
- [7] D. W. Decker and C. T. Kelley. Convergence acceleration for Newton’s method at singular points. *SIAM Journal on Numerical Analysis*, 19:219–229, 1982.

- [8] Jean Pierre Dedieu and Mike Shub. On simple double zeros and badly conditioned zeros of analytic functions of n variables. *Mathematics of Computation*, 70(233):319–327, 2001.
- [9] M. Giusti, G. Lecerf, B. Salvy, and J.-C. Yakoubsohn. On location and approximation of clusters of zeros of analytic functions. *Foundations of Computational Mathematics*, 5(3):257–311, 2005.
- [10] M. Giusti, G. Lecerf, B. Salvy, and J.-C. Yakoubsohn. On location and approximation of clusters of zeros: case of embedding dimension one. *Foundations of Computational Mathematics*, 7(1):1–58, 2007.
- [11] M. Giusti and J.-C. Yakoubsohn. Multiplicity hunting and approximating multiple roots of polynomial systems. In *Contemporary Mathematics 604, Recent Advances in Real Complexity and Computation*, pages 104–128. American Mathematical Society, 2013.
- [12] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.
- [13] G. Golub and C. Van Loan. *Matrix computations*. Johns Hopkins University Press, 3rd edition, 1996.
- [14] Andreas Griewank. *Analysis and modification of Newton’s method at singularities*. Thesis, Australian National University, 1980.
- [15] Andreas Griewank and M. R. Osborne. Newton’s method for singular problems when the dimension of the null space is > 1 . *SIAM Journal on Numerical Analysis*, 18:145–149, 1981.
- [16] Jonathan D. Hauenstein, Bernard Mourrain, and Agnes Szanto. Certifying isolated singular points and their multiplicity structure. In *Proceedings of the 2015 ACM on International Symposium on Symbolic and Algebraic Computation*, ISSAC ’15, pages 213–220, New York, NY, USA, 2015. ACM.
- [17] Jonathan D. Hauenstein and Frank Sottile. Algorithm 921: AlphaCertified: Certifying solutions to polynomial systems. *ACM Trans. Math. Softw.*, 38(4):28:1–28:20, August 2012.
- [18] Jonathan D. Hauenstein and Charles W. Wampler. Isosingular sets and deflation. *Foundations of Computational Mathematics*, 13(3):371–403, 2013.
- [19] Erich L. Kaltofen, Bin Li, Zhengfeng Yang, and Lihong Zhi. Exact certification in global polynomial optimization via sums-of-squares of rational functions with rational coefficients. *Journal of Symbolic Computation*, 47(1):1–15, 2012. In memory of Wenda Wu (1929–2009).
- [20] G. Lecerf. Quadratic Newton iteration for systems with multiplicity. *Foundations of Computational Mathematics*, 2(3):247–293, 2002.
- [21] Anton Leykin, Jan Verschelde, and Ailing Zhao. Newton’s method with deflation for isolated singularities of polynomial systems. *Theoretical Computer Science*, 359(1):111–122, 2006.
- [22] Anton Leykin, Jan Verschelde, and Ailing Zhao. Higher-order deflation for polynomial systems with isolated singular solutions. In Alicia Dickenstein, Frank-Olaf Schreyer, and Andrew J. Sommese, editors, *Algorithms in Algebraic Geometry*, volume 146 of *The IMA Volumes in Mathematics and its Applications*, pages 79–97. Springer New York, 2008.
- [23] Nan Li and Lihong Zhi. Compute the multiplicity structure of an isolated singular solution: case of breadth one. *Journal of Symbolic Computation*, 47:700–710, 2012.
- [24] Nan Li and Lihong Zhi. Computing isolated singular solutions of polynomial systems: case of breadth one. *SIAM Journal on Numerical Analysis*, 50(1):354–372, 2012.
- [25] Nan Li and Lihong Zhi. Verified error bounds for isolated singular solutions of polynomial systems: case of breadth one. *Theoretical Computer Science*, 479:163–173, 2013.
- [26] Nan Li and Lihong Zhi. Verified error bounds for isolated singular solutions of polynomial systems. *SIAM Journal on Numerical Analysis*, 52(4):1623–1640, 2014.
- [27] Angelos Mantzaflaris and Bernard Mourrain. Deflation and certified isolation of singular zeros of polynomial systems. In A. Leykin, editor, *Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, ISSAC ’11, pages 249–256, New York, NY, USA, 2011. ACM.
- [28] Maria Grazia Marinari, Teo Mora, and Hans Michael Möller. Gröbner duality and multiplicities in polynomial system solving. In *Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation*, ISSAC ’95, pages 167–179, New York, NY, USA, 1995. ACM.
- [29] Alexander P Morgan, Andrew J Sommese, and Charles W Wampler. Computing singular solutions to polynomial systems. *Advances in Applied Mathematics*, 13(3):305–327, 1992.

- [30] B. Mourrain. Isolated points, duality and residues. *J. of Pure and Applied Algebra*, 117 & 118:469–493, 1996.
- [31] Takeo Ojika. Modified deflation algorithm for the solution of singular problems. i. a system of nonlinear algebraic equations. *Journal of Mathematical Analysis and Applications*, 123(1):199 – 221, 1987.
- [32] Takeo Ojika, Satoshi Watanabe, and Taketomo Mitsui. Deflation algorithm for the multiple roots of a system of nonlinear equations. *Journal of Mathematical Analysis and Applications*, 96(2):463 – 479, 1983.
- [33] L.B. Rall. Convergence of the Newton process to multiple solutions. *Numerische Mathematik*, 9(1):23–37, 1966.
- [34] G. W. Reddien. On Newton’s method for singular problems. *SIAM Journal on Numerical Analysis*, 15(5):993–996, 1978.
- [35] G. W. Reddien. Newton’s method and high order singularities. *Computers & Mathematics with Applications*, 5(2):79 – 86, 1979.
- [36] Siegfried M. Rump and Stef Graillat. Verified error bounds for multiple roots of systems of nonlinear equations. *Numerical Algorithms*, 54(3):359–377, 2010.
- [37] Michael Shub and Steve Smale. Complexity of bezout’s theorem IV: Probability of success; extensions. *SIAM Journal on Numerical Analysis*, 33(1):128–148, 1996.
- [38] Mike Shub and Steve Smale. Computational complexity: On the geometry of polynomials and a theory of cost: I. *Annales Scientifiques De L École Normale Supérieure*, 18(1):107–142, 1985.
- [39] Mike Shub and Steve Smale. Computational complexity: On the geometry of polynomials and a theory of cost: II. *SIAM Journal on Computing*, 15(1):145–161, 1986.
- [40] Steve Smale. The fundamental theorem of algebra and complexity theory. *Bulletin of the American Mathematical Society*, 4(1):1–36, 1981.
- [41] Steve Smale. Newton’s method estimates from data at one point. In *The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics*, pages 185–196. Springer New York, 1986.
- [42] H. Stetter. *Numerical Polynomial Algebra*. SIAM, Philadelphia, 2004.
- [43] G. W. Stewart. Error and perturbation bounds for subspaces associated with certain eigenvalue problems. *SIAM Review*, 15(4):727–764, 1973.
- [44] Xinghua Wang and Danfu Han. On dominating sequence method in the point estimate and smale theorem. *Science in China Ser A*, 33(2):135–144, 1990.
- [45] Xiaoli Wu and Lihong Zhi. Determining singular solutions of polynomial systems via symbolic-numeric reduction to geometric involutive forms. *Journal of Symbolic Computation*, 47(3):227–238, 2012.
- [46] Jean Claude Yakoubsohn. Finding a cluster of zeros of univariate polynomials. *Journal of Complexity*, 16(3):603–638, 2000.
- [47] Jean-Claude Yakoubsohn. Simultaneous computation of all zero clusters of an univariate polynomial. In *Foundations of Computational Mathematics*, pages 433–455. World Sci. Publishing, 2002.
- [48] Norio Yamamoto. Regularization of solutions of nonlinear equations with singular jacobian matrices. *Journal of Information Processing*, 7(1):16–21, March 1984.

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